Potato Chip Shape Optimization

Due Thursday, March 3 at 11:59 p.m. on D2L

You must read all instructions on the lab course webpage in the Projects section, in addition to these instructions (beginning to end!). Failure to do so may result in a loss of points, or possibly no credit for the lab.

1 What is Shape Optimization?

An engineer is always concerned with whether or not their design will work. More importantly, a *good* engineer is constantly questioning if their design is the *right* design. First consider the following example:

An aerospace engineer is designing a communications satellite for geosynchronous orbit. During the design phase, the engineer will ponder top-level questions such as "what will be the satellite's power source?" and "what type of propellent and rocket engine should be used for attitude control and stabilization?" However, to successfully answer these questions, the engineer must appropriately *size* the satellite by answering the questions, "how much power will the satellite need?" and "how frequently do we expect to update the satellite's orientation?" Moreover, these questions bring up more regarding structural and avionics considerations. At the end of the day, the final design will *not* be the most powerful satellite, nor the most stable or sturdiest satellite. Rather, a design which is optimized for its specific application, making power and structure compromises for overall mass reduction.

In this project, you will hold a different, yet equally important, position of a "chip engineer" who is tasked with optimizing the geometry of a potato chip under geometrical, material, and monetary constraints. In particular, the end result of the chip will resemble a Pringle® (See Figure 1 below) which we will achieve through the concept of shape optimization.

Figure 1: Pringles® potato chips - a hyperbolic paraboloid

2 Container Design

Your first task is to design the container for the chips. The stores at which you plan to sell this chip product have informed you that they can provide you with a shelf that is 21" wide, 15" deep, and 12" tall to place your product for sale. Given these constraints, you have deduced that the height of your container should be 10", providing a 2" gap for ease of the customer to remove the box from the shelf. Additionally, you've determined that the container should be a cylinder with circular cross-section; this choice will maximize the number of containers that can fit on a shelf. The cardboard manufacturer, who will provide you with the material to make containers, has given you the following equation which is their estimate of the price in dollars of 100 containers for any given diameter d:

$$P_R(d) = 0.8e^{-7(d-3.5)^2} - 0.6e^{-25(d-2.25)^2} - 0.5e^{-20(d-1.5)^2} + 1.1\log(d+2) \tag{1}$$

Plot equation (1) against d for $d \in [0,4]$ and determine **all** the local extrema. Tabulate the results and deduce which value(s) are candidates for the final chip radius. Additionally, create a plot for each of the candidate containers (you don't need to plot the top and bottom of the cylinders).

3 Chip Geometry

The geometry of a hyperbolic paraboloid (Pringle®) is desirable since it has a circular cross-section but has a larger surface area resulting in an overall better chip eating experience for the customer. A function $f(x,y)$ in the *Pringle family of hyperbolic paraboloids* takes the form $f(x,y) = \beta y^2 - \alpha x^2$ for $\alpha, \beta \in \mathbb{R}$. Our task now is to optimize the chip geometry by minimizing price. Given the current cost of chip ingredients, a fellow chip engineer provides you with the following function which represents the price in dollars of 100 chips as a function of the Pringle parameters α, β:

$$P_G(\alpha,\beta) = 1 - \frac{\sqrt{3}}{2}e^{-90\left((\alpha-0.159)^2+(\beta-.341)^2\right)} - 0.68e^{-95\left((\alpha-0.242)^2+(\beta-0.1)^2\right)} \tag{2}$$

Provide both a 3 dimensional plot and a contour plot of $P_G(\alpha,\beta)$ on the range $0 \le \alpha, \beta \le 0.5$ and from the local extrema, determine the candidate geometries for the final chip design.

Plot the candidate chips corresponding to the candidate geometries. Note that the region the chip surface $f(x,y)$ should be plotted over a disk with diameter equal to the container diameter. You can restrict the domain of a 3D plot through the option `RegionFunction`. Be sure to set the `BoxRatios` option to `Automatic` to ensure the plot axes are scaled uniformly.

4 Chip Thickness

Consumer reports have shown that customers are the most satisfied with containers with 95 chips. Determine the thickness of the chip, for each of the candidate chips found above, to fit 95 chips in a container. You will need the height of each chip which is the difference between the maximum and minimum values of the chip's boundary. Use Lagrange multipliers to find these boundary extrema via the `FindRoot` function. **Note that you must use Lagrange Multipliers here to receive credit!** Tabulate the results and draw conclusions on whether or not any candidate chips can be eliminated. The same consumer reports as before have also shown that customers find any chip thicker than 0.1" too thick, therefore you should remove any chips above this threshold from consideration.

5 The 'Wow' Factor

After a very careful study, an equation was developed relating the maximum principal curvatures of the chip and the chip surface area to customer satisfaction. In the case of this chip, the first principal curvature κ_1 is the maximum curvature of the chip along the x-axis direction and the second principal curvature κ_2 is the maximum curvature of the chip along the y-axis direction. The equation is classically referred to as the "Wow Factor" and is given by

$$WF(\kappa_1, \kappa_2, SA) = \sin\left(\frac{\kappa_1 \pi}{.864}\right) + e^{-(\kappa_2+.432)^2(\kappa_2-.432)^2} + \frac{4}{SA - 8\pi}$$

where larger values of WF correspond to chips that are more appealing to customers. Compute the maximum principal curvatures κ_1, κ_2 and surface areas SA. Tabulate them along with their corresponding "Wow Factors" for each of the remaining candidate geometries, and then then determine the optimal chip to manufacture and sell.

The surface area of a candidate chip with diameter d and Pringle parameters α, β can be computed with the following Mathematica function:

```
SA[alpha_, beta_, d_] :=
 NIntegrate[
  Boole[-d/2 <= x <= d/2 && -Sqrt[(d/2)^2 - x^2] <= y <= Sqrt[(d/2)^2 - x^2]]*
   Sqrt[4*alpha^2*x^2 + 4*beta^2*y^2 + 1], {x, -d/2, d/2}, {y, -d/2, d/2}];
```

Hint: For your chips, you can compute the maximum principal curvatures by parameterizing the x- and y-axes, project them onto the chip (i.e. find $z(t)$), and then find the maximum curvature of the resulting plane curves.

6 Chips at a Picnic

You and your friend are having a picnic at Chautauqua, eating your newly-developed chips. Your friend drops a chip on the ground and you see an ant is crawling on the chip. Before you blow the ant off and proceed to eat the chip, you notice the ant is walking along the path in the xy-plane described by

$$\mathbf{r}(t) = \frac{1}{10}\langle -2\cos t + 5\cos\left(\frac{t}{5}\right), -2\sin t + 7\sin\left(\frac{t}{5}\right)\rangle, \quad 0 \le t < 10\pi$$

Plot this path on the chip with a black sphere denoting where you first saw the ant at $t = \pi/7$. Plot the slope of the path for all t on which the ant is walking (Note this is not the same as the gradient of the potato chip along the path!). What is the slope of the path at the moment you first noticed the ant?