1. (28pts) The following problems are not related.

(a)(12pts) Suppose we know that the function $f(x)$ is an *even* function. Show that the function $g(x) = \sin(x) + xf(x)$ is an *odd* function. Justify your answer.

(b)(12pts) Evaluate the limit: $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{x^2+x}\right)$

(c)(4pts) The function $h(x) = \frac{3x+1}{\sqrt[3]{8x^3+5}}$ has a *horizontal asymptote* at which choice below? (**No justification necessary** - *Choose only one answer, copy down the entire answer.*)

(A) $y=0$ (B) $y=\frac{3}{2}$ (C) $y=0$ and $y=3/2$ (D) $y=-3/2$ and $y=3/2$ (E) None of these

Solution: (a)(12pts) Note that since $f(x)$ is even we have that $f(-x) = f(x)$ and recall that $\sin(x)$ is an odd function so $\sin(-x) = -\sin(x)$ thus

$$g(-x) = \sin(-x) + (-x)f(-x) = -\sin(x) - xf(x) = -\big(\sin(x) + xf(x)\big) = -g(x)$$

thus $g(-x) = -g(x)$ and so $g(x)$ is an odd function.

(b)(12pts) Note that if we apply the limit directly, we get the indeterminate form "$\infty - \infty$" and combining the fractions and applying the limit yields

$$\lim_{x\to 0^+}\frac{1}{x} - \frac{1}{x^2+x} = \lim_{x\to 0^+}\frac{(x+1)-1}{x(x+1)} = \lim_{x\to 0^+}\frac{x}{x(x+1)} = \lim_{x\to 0^+}\frac{1}{x+1} = 1.$$

(c)(4pts) Choice (B). Discussion: Note that

$$\lim_{x\to\infty}\frac{3x+1}{\sqrt[3]{8x^3+5}} = \lim_{x\to\infty}\frac{x\cdot(3+1/x)}{x\cdot\sqrt[3]{8+5/x^3}} = \frac{3}{\sqrt[3]{8}} = \frac{3}{2} \text{ and } \lim_{x\to-\infty}\frac{3x+1}{\sqrt[3]{8x^3+5}} \overset{DOP}{=} \frac{3}{2}$$

thus we see that $y = 3/2$ is the only horizontal asymptote which implies choice (B).

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2. (24pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) Use the *Squeeze Theorem* to evaluate the following limit: $\lim_{x\to 0^+}\sqrt{x}\cos^2\left(\frac{1}{x}\right)$. Show all work, explain your answer.

(b)(12pts) Find the limit $\lim_{x\to 0}\frac{\sin(3x)\sin(5x)}{x^2}$. Justify your answer, show all work.

Solution: (a)(12pts) Note that for all $x > 0$ we have

$$-1 \le \cos(1/x) \le 1 \;\Rightarrow\; 0 \le \cos^2(1/x) \le 1 \;\Rightarrow\; 0 \le \sqrt{x}\cos^2(1/x) \le \sqrt{x}$$

and, finally, observe that $\lim_{x\to 0^+} 0 = \lim_{x\to 0^+}\sqrt{x} = 0$, thus, by Squeeze Theorem, we have $\lim_{x\to 0^+}\sqrt{x}\cos^2(1/x) = 0$.

(b)(12pts) Using the "special" limit $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$ we have

$$\lim_{x\to 0}\frac{\sin(3x)\sin(5x)}{x^2}=\lim_{x\to 0}\frac{\sin(3x)}{x}\cdot\frac{\sin(5x)}{x}=\lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\frac{\sin(5x)}{5x}\cdot 3\cdot 5=1\cdot 1\cdot 15=\boxed{15}$$

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3. (28pts) Start this problem on a **new** page. The following problems are not related.

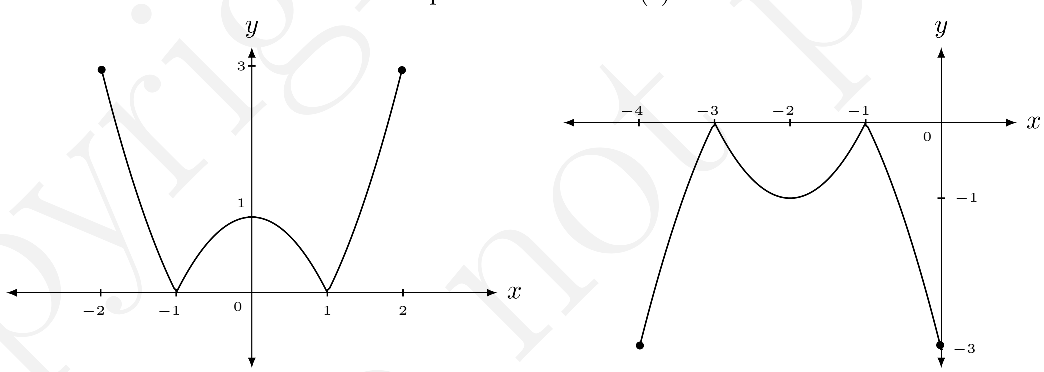
(a)(12pts) Evaluate the limit: $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}$. Show all work.

(b)(12pts) Suppose $g(x)=\begin{cases} x^2+x, & \text{if } x<0,\\ 1-\cos(x), & \text{if } x=0,\\ \sin(x), & \text{if } x>0.\end{cases}$ (i)(6pts) Find the $\lim_{x\to 0} g(x)$. (ii)(6pts) Show that $g(x)$ is continuous at $x=0$. Be sure to show that all the conditions of continuity have been satisfied.

(c)(4pts) Consider the graph of the function below labeled as Graph A. If this function is $y=f(x)$ then which of the following choices given below correctly represents the graph labeled as Graph B? **No justification necessary** - *Choose only* *<u>one</u>* *answer, copy down the entire answer.)*

(A) $y=-f(x)-2$ (B) $y=f(-x+2)$ (C) $y=f(-x)-2$ (D) $y=-f(x+2)$ (E) $y=f(-x)+2$

Graphs for Problem 3(c)



(a) Graph A – the function $y=f(x)$ (b) Graph B

Solution: (a)(12pts) Multiplying by the conjugate yields

$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}=\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}=\lim_{x\to 0}\frac{(3+x)-3}{x(\sqrt{3+x}+\sqrt{3})}=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{3+x}+\sqrt{3})}=\boxed{\frac{1}{2\sqrt{3}}}.$$

(b)(i)(6pts) At $x=0$ we have to check the one-side limits, note that

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-} x^2+x=0 \text{ and } \lim_{x\to 0^+} g(x)=\sin(0)=0 \Rightarrow \lim_{x\to 0} g(x)=0.$$

(b)(ii)(6pts) <u>Yes, $g(x)$ is continuous</u> since $g(0)=1-\cos(0)=1-1=0$ so $\lim_{x\to 0^-} g(x)=\lim_{x\to 0^+} g(x)=0=g(0)$.

(c)(4pts) $\boxed{\text{Choice D.}}$ If we shift Graph A to the left 2 units and then reflect it about the x-axis this yields Graph B.

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4. (20pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) Use limits to classify all discontinuities of: $f(x) = \dfrac{x-2}{x^3-2x^2}$. Justify with limits.

(b)(8pts) The function $g(x) = \dfrac{x+10}{|x|+2}$ has two horizontal asymptotes. They are $y = 1$ and $y = -1$. Use the Intermediate Value Theorem to show that $g(x)$ crosses one of its horizontal asymptotes on the interval $[-10, 0]$. Clearly explain your answer.

Solution: (a)(12pts) Factoring the denominator yields $x^3 - 2x^2 = x^2(x-2)$, and so we see that there are possible vertical asymptotes at $x = 0, 2$. Taking the limit as $x \to 2$ yields the indeterminate form "0/0", and factoring yields

$$\lim_{x\to 2} \frac{x-2}{x^3-2x^2} \overset{\text{"0/0"}}{=} \lim_{x\to 2} \frac{x-2}{x^2(x-2)} = \lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4}$$

thus we have a *removable discontinuity* at $x = 2$. Now, taking limits as $x \to 0$ gives

$$\lim_{x\to 0} \frac{x-2}{x^3-2x^2} = \lim_{x\to 0} \frac{x-2}{x^2(x-2)} = \lim_{x\to 0} \frac{1}{x^2} \overset{N/0}{=} +\infty$$

thus we have a *infinite discontinuity* (and vertical asymptote) at $x = 0$.

(b)(8pts) Note that $g(-10) = 0$ and $g(0) = \frac{10}{2} = 5$ and since $g(x)$ is a continuous function for all x and since $g(-10) \le 1 \le g(0)$, by the Intermediate Value Theorem, there exists some number c in $(-10, 0)$ such that $g(c) = 1$ thus we see that $g(x)$ crosses one of its horizontal asymptotes.