APPM 1340
Exam 1
Fall 2024

Name		
Instructor	Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

There is a **FORMULA SHEET** on the **LAST PAGE** of this exam

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

1. (21 pts) Parts (a) - (d) are not related.

   (a) Identify the domain of the function $f(x) = \dfrac{x^2 - x + 3}{\sqrt{4 - x^2}}$. Express your answer using interval notation.

   Solution:

   In order to avoid taking the square root of a negative number, we must have $4 - x^2 \geq 0$.

$$0 \leq 4 - x^2$$

$$x^2 \leq 4$$

$$|x| \leq \sqrt{4} = 2$$

$$-2 \leq x \leq 2$$

   In order to avoid division by zero, we must have $4 - x^2 \neq 0$, so that $x \neq \pm 2$.

   Therefore, $-2 < x < 2$, which is expressed as $\boxed{(-2, 2)}$ using interval notation.

   (b) The function $\dfrac{x-1}{\left(\dfrac{x-2}{x+1}\right)}$ has been expressed as a complex fraction.

      i. Identify the domain of the function. You do **not** need to express your answer using interval notation.

      Solution:

      If $x = -1$, the denominator of the complex fraction would involve division by zero.

      If $x = 2$, the denominator of the complex fraction would equal zero, which would lead to division by zero when evaluating the overall complex fraction.

      Therefore, the domain of $g(x)$ is $\boxed{x \neq -1, x \neq 2}$

      Alternatively, in interval form, the domain is $\boxed{(-\infty, -1) \cup (-1, 2) \cup (2, \infty)}$

      ii. Simplify the expression for $g(x)$ by replacing the complex fraction with a rational expression (polynomial over polynomial).

      Solution:

      Division by a fraction is equivalent to multiplication by the reciprocal of that fraction.

$$\frac{x-1}{\left(\dfrac{x-2}{x+1}\right)} = (x-1) \cdot \left(\frac{x+1}{x-2}\right) = \boxed{\frac{(x-1)(x+1)}{x-2} = \frac{x^2-1}{x-2}}$$

(c) Solve the inequality $-4 \le -3x + 5 < 11$. Express your answer using interval notation.

Solution:

$$-4 \le -3x + 5 < 11$$

$$-9 \le -3x < 6$$

$$3 \ge x > -2 \qquad \text{(Dividing all terms by } -3 \text{ causes the direction of each inequality to reverse)}$$

Therefore, using interval notation, the solution is $\boxed{(-2, 3]}$

(d) Find all solutions, if any, of the equation $x^{8/3} = 8x^{2/3} - 2x^{5/3}$.

Solution:

$$x^{8/3} = 8x^{2/3} - 2x^{5/3}$$

$$x^{8/3} + 2x^{5/3} - 8x^{2/3} = 0$$

$$x^{2/3}(x^2 + 2x - 8) = 0$$

$$x^{2/3}(x - 2)(x + 4) = 0$$

Any value of x for which any of the preceding three factors equals zero is a solution to the original equation.

$$x^{2/3} = 0 \;\Rightarrow\; x = 0$$

$$x - 2 = 0 \;\Rightarrow\; x = 2$$

$$x + 4 = 0 \;\Rightarrow\; x = -4$$

Therefore, the set of solutions is $\boxed{x = -4, 0, 2}$

2. (29 pts) Parts (a) and (b) are not related.

   (a) For parts i-iii, let point A be $(1, 2)$, let point B be $(3, -3)$, let segment AB be the line segment connecting points A and B, and let point M be the midpoint of segment AB.

      i. Find the (x, y) coordinates of point M.

      Solution:

      The general expression for the midpoint of the line segment connecting the points (x_1, y_1) and (x_2, y_2) is

      $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

      Substituting the coordinate values of points A and B leads to the following midpoint of segment AB:

      $$\left(\frac{1+3}{2}, \frac{2-3}{2}\right) = \boxed{\left(2, -\frac{1}{2}\right)}$$

      ii. Find the length of segment AB.

      Solution:

      The length of segment AB is the distance between points A and B, so we'll apply the distance formula.

      $$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3-1)^2 + (-3-2)^2} = \sqrt{2^2 + (-5)^2}$$

      $$D = \boxed{\sqrt{29}}$$

      iii. Find an equation of the line that is perpendicular to segment AB and passes through point A.

      Solution:

      The first step is to determine the slope of segment AB. We'll let m_1 denote that slope.

      $$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{3-1} = \frac{-5}{2} = -\frac{5}{2}$$

      The slopes of two perpendicular lines are negative reciprocals of each other. Since the slope of segment AB is $-5/2$, the slope of a line that is perpendicular to segment AB is $2/5$.

      The point-slope form of the equation of the line passing through a point (x_0, y_0) is $y - y_0 = m(x - x_0)$.

      Since point A lies on the line that is perpendicular to segment AB at point A, we'll use $(x_0, y_0) = (1, 2)$, which are the coordinates of point A.

      Therefore, an equation for the line that is perpendicular to segment AB and passes through point A is

      $$\boxed{y - 2 = \frac{2}{5}(x - 1)}$$

(b) Find the center and radius of the circle whose equation is $x^2 - 8x + 16 + y^2 + 2y + 1 = 7$.

Solution:

The equation of a circle centered at a point (h, k) with a radius of r is $(x - h)^2 + (y - k)^2 = r^2$.

The expression $x^2 - 8x + 16$ can be expressed in factored form as $(x - 4)(x - 4) = (x - 4)^2$.

Similarly, $y^2 + 2y + 1$ can be expressed in factored form as $(y + 1)(y + 1) = (y + 1)^2$.

Therefore, the original equation for the given circle can be expressed as follows, which corresponds to the general form of an equation of a circle:

$$x^2 - 8x + 16 + y^2 + 2y + 1 = 7$$

$$(x - 4)^2 + (y + 1)^2 = 7$$

$$(x - 4)^2 + (y - (-1))^2 = (\sqrt{7})^2$$

Comparing this result to the general form of an equation of a circle, we see that $h = 4$, $k = -1$, and $r = \sqrt{7}$. Therefore, the center and radius of the given circle are as follows:

Center: $\boxed{(4, -1)}$

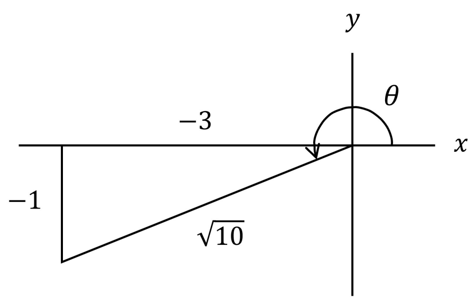
Radius: $\boxed{\sqrt{7}}$

3. (28 pts) Parts (a) - (c) are not related.

(a) If $\cot\theta = 3$ and θ is on the interval $(\pi, 3\pi/2)$, find the value of $\cos\theta$.

Solution:

Since θ is on the interval $(\pi, 3\pi/2)$, there is a triangle in quadrant III that is associated with angle θ, as shown in the following figure.



Since the triangle is in quadrant III, both the *adjacent* (horizontal) and *opposite* (vertical) components of the triangle are negative. Since $\cot\theta = 3$, the ratio of the *adjacent* component to the *opposite* component is 3. Therefore, we can let the *adjacent* component equal -3 and the *opposite* component equal -1 (since both are negative, and $-3/-1 = 3$), as depicted in the preceding figure.

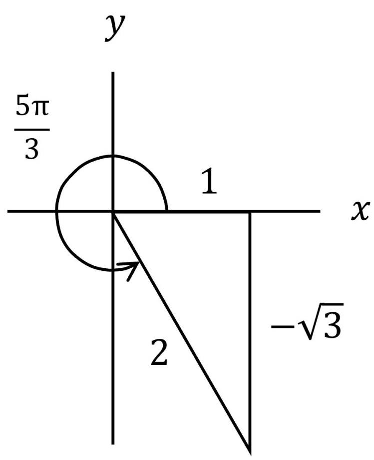
The length of the hypotenuse of the triangle can be determined to be $\sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$, per the Pythagorean Theorem. That result is also depicted in the figure.

Therefore, $\cos\theta = \dfrac{\textit{adjacent}}{\textit{hypotenuse}} = \dfrac{-3}{\sqrt{10}} = \boxed{-\dfrac{3}{\sqrt{10}}}$

(b) Evaluate $\sin\left(\dfrac{5\pi}{3}\right)$

Solution:

Since $\dfrac{3\pi}{2} < \dfrac{5\pi}{3} < 2\pi$, the angle $\dfrac{5\pi}{3}$ lies in Quadrant IV, as drawn in the following figure.



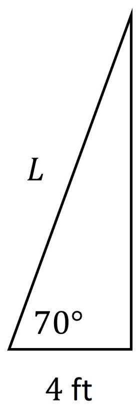
The reference angle is $2\pi - \dfrac{5\pi}{3} = \dfrac{\pi}{3}$, which is a $60°$ angle in the special $30° - 60° - 90°$ right triangle. The dimensions of such a triangle are proportional to 1, $\sqrt{3}$, and 2, which leads to the set of dimensions displayed in the figure, with the negative value of the *opposite* component of the triangle accounting for the fact that the triangle is in Quadrant IV.

It follows from the figure that $\sin\left(\dfrac{5\pi}{3}\right) = \dfrac{\textit{opposite}}{\textit{hypotenuse}} = \dfrac{-\sqrt{3}}{2} = \boxed{-\dfrac{\sqrt{3}}{2}}$

(c) A ladder is leaning against a vertical wall. Suppose the ladder and the floor form a 70° angle and the bottom of the ladder is 4 feet from the base of the wall. Express the length L of the ladder in terms of the given measurements. Include the correct unit of measurement. Your answer should include a trigonometric function.

Solution:

The situation is depicted in the following figure, with the horizontal leg representing the floor, the vertical leg representing the wall, and the hypotenuse representing the ladder:



According to the figure, $\cos 70^\circ = \dfrac{\textit{adjacent}}{\textit{hypotenuse}} = \dfrac{4}{L}$.

Therefore, $\boxed{L = \dfrac{4}{\cos 70^\circ}\ \text{ft}}$

4. (22 pts) Parts (a) - (c) are not related.

(a) Find all values of x in the interval $[0, 2\pi]$ that satisfy the following equation:

$$(\cos x)\left(2\cos^2 x - 1\right) = 0$$

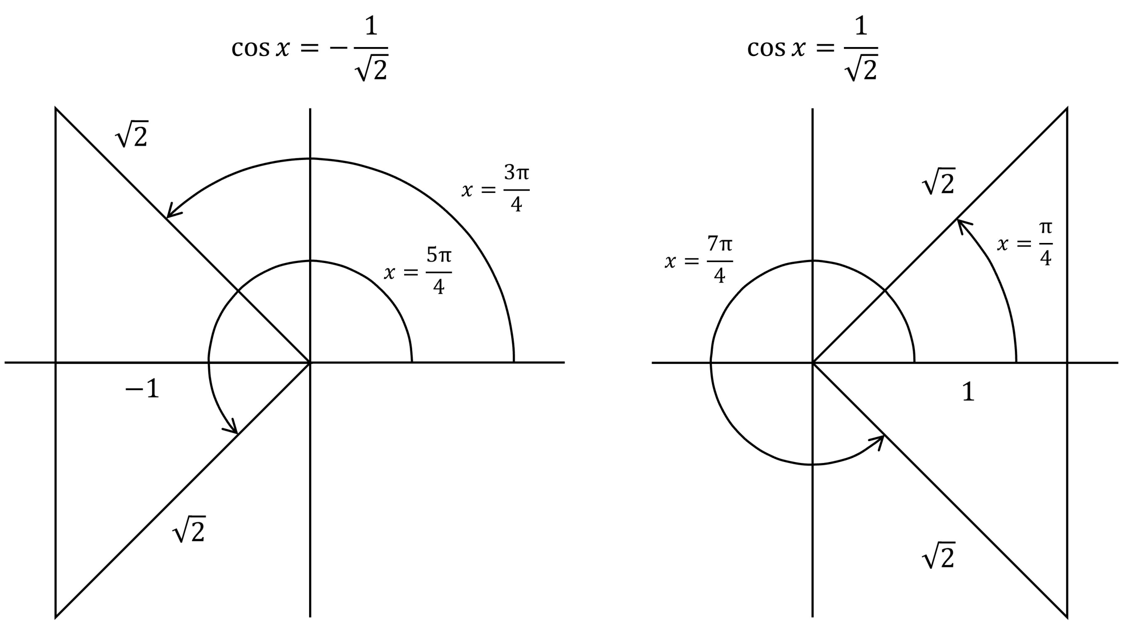
Solution:

The Zero Factor Theorem indicates that $\cos x = 0$ or $2\cos^2 x - 1 = 0$.

$$\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$2\cos^2 x - 1 = 0 \quad \Rightarrow \quad \cos^2 x = \frac{1}{2} \quad \Rightarrow \quad \cos x = \pm\frac{1}{\sqrt{2}}$$

The following figures can assist in evaluating $\cos x = \pm\frac{1}{\sqrt{2}}$:



Both triangles in the left-hand figure are associated with an angle whose cosine is $-1/\sqrt{2}$ and both triangles in the right-hand figure are associated with an angle whose cosine is $1/\sqrt{2}$. The leg of length 1 and the hypotenuse of length $\sqrt{2}$ together imply that both triangles are special $45° - 45° - 90°$ right triangles. In each such triangle, the acute angles are $\pi/4$ radians, which is the reference angle for each of the four depicted triangles. Therefore, there are four solutions to the equation $\cos x = \pm 1/\sqrt{2}$:

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Combining the preceding result with the solutions to the equation $\cos x = 0$ found earlier yields the following six solutions to the original equation:

$$\boxed{x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}}$$

(b) A circular sector of radius 3 has an area of $3\pi/5$. What is the central angle of the sector? Express your answer in **degrees**.

Solution:

Accoring to the formula sheet on the final page of the exam, $A = \frac{1}{2}\theta r^2$.

$$\frac{3\pi}{5} = \frac{1}{2}(\theta)(3^2)$$

$$\left(\frac{3\pi}{5}\right)\left(\frac{2}{9}\right) = \theta$$

$$\theta = \frac{6\pi}{45} = \frac{2\pi}{15}$$

Therefore, $\theta = \left(\frac{2\pi}{15} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right) = \boxed{24°}$

(c) Is the function $h(x) = (x^2 + 1)^3$ odd, even, or neither? Justify your answer by using the definition of odd and/or even functions.

Solution:

$h(-x) = ((-x)^2 + 1)^3 = (x^2 + 1)^3 = h(x)$

Since $h(-x) = h(x)$, $h(x)$ is an $\boxed{\text{even function}}$

Potentially Useful Formulas

Sector of a circle:

Arc length: $L = \theta r$

Area: $A = \frac{1}{2}\theta r^2$

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Sums and differences:

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Double-angle formulas:

$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$