1. (38 pts) Decide whether the following expressions are convergent or divergent. If convergent, find the value the expression converges to. Explain your reasoning and name any test you use.

   (a) The sequence given by $a_n = \dfrac{5 \cdot 10 \cdot 15 \cdots (5n)}{n!}$ for $n = 1, 2, 3, \ldots$

   (b) $\displaystyle\sum_{k=2}^{\infty} [\ln(k/2) - \ln(k)]$

   (c) $\displaystyle\sum_{n=1}^{\infty} (\cos(3))^n$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$

Solution:

(a) $a_n = \dfrac{5 \cdot 10 \cdot 15 \cdots (5n)}{n!} = \dfrac{(5^n)n!}{n!} = 5^n$. Therefore, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5^n = \infty$ and the sequence $\boxed{\text{diverges}}$.

(b) First note that $\ln(k/2) - \ln(k) = \ln(1/2) = -\ln(2)$. Therefore, the series $\displaystyle\sum_{k=2}^{\infty} [\ln(k/2) - \ln(k)]$ $\boxed{\text{diverges}}$ by the Test for Divergence since $\displaystyle\lim_{k\to\infty} [\ln(k/2) - \ln(k)] = \ln(1/2) = -\ln(2) \neq 0$.

(c) $\displaystyle\sum_{n=1}^{\infty} (\cos(3))^n$ is a geometric series with $a = \cos(3)$ and common ratio $r = \cos(3)$. Since $-1 < \cos(3) < 1$, the series $\boxed{\text{converges}}$ and $\displaystyle\sum_{n=1}^{\infty} (\cos(3))^n = \frac{\cos(3)}{1 - \cos(3)}$

(d) Use the integral test. Let $f(x) = \dfrac{1}{x(\ln(x))^{1/2}}$. For $x \geq 2$, we have that f is positive and continuous. It is also decreasing since the denominator gets larger as n increases. (We could also calculate $f'(x) = -\dfrac{(\ln(x))^{1/2} + 1/\left(2(\ln(x))^{1/2}\right)}{x^2 \ln(x)}$, which is always negative.). We now consider

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln(x))^{1/2}}\,dx &= \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln(x))^{1/2}}\,dx \\
&= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u^{1/2}}\,dx \quad \text{substitute } u = \ln(x) \text{ and } du = (1/x)dx \\
&= \lim_{b\to\infty} 2u^{1/2} \Big|_{\ln(2)}^{\ln(b)} \\
&= \lim_{b\to\infty} \left[2(\ln(b))^{1/2} - 2(\ln(2))^{1/2}\right] = \infty
\end{aligned}$$

Since the integral diverges, we know that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$ also diverges.

2. (26 pts) Let $f(x) = e^{-x/3}$.

(a) Find a power series representation for f centered at $a = 3$.

(b) Find T_2, the second order Taylor polynomial for f, centered at $a = 3$.

(c) Use the Taylor Remainder Formula to find an error bound if T_2 is used to approximate f for x in the interval $[3, 3.3]$. Express your answer in terms of e.

Solution:

(a) We begin by computing the derivatives of f and evaluating them at $a = 3$:

$$\begin{aligned} f(x) &= e^{-x/3} & f(3) &= e^{-1} \\ f'(x) &= (-1/3)e^{-x/3} & f'(3) &= (-1/3)e^{-1} \\ f''(x) &= (-1/3)^2 e^{(-x/3)} & f''(3) &= (-1/3)^2 e^{-1} \\ &\dots & &\dots \\ f^{(n)}(x) &= (-1/3)^n e^{(-x/3)} & f^{(n)}(3) &= (-1/3)^n e^{-1} = \frac{(-1)^n}{3^n e} \end{aligned}$$

We then have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n! e}(x-3)^n}$$

(b) $\boxed{T_2(x) = \frac{1}{e} - \frac{1}{3e}(x-3) + \frac{1}{3^2 2! e}(x-3)^2}$

(c) We have $R_2(x) = \frac{f^{(3)}(z)}{3!}(x-3)^3$ where z is between x and $a = 3$. Since our x values are in the interval $[3, 3.3]$, we need to find the maximum of $|f^{(3)}(z)|$ for z in $(3, 3.3)$. Thus,

$$|f^{(3)}(z)| = |(-1/3)^3 e^{(-z/3)}| < (1/27)e^{-1}$$

This gives

$$|R_2(x)| = \left| \frac{f^{(3)}(z)}{3!}(x-3)^3 \right| < \frac{(.3)^3}{27 \cdot 3! \cdot e} = \boxed{\frac{1}{6000e}}$$

3. (28 pts) The following two problems are not related.

   (a) Evaluate $\displaystyle\int x\sec^2 x\,dx$

   (b) i. Find the first 3 nonzero terms of the Maclaurin series for the function $\sqrt{1+x}$.

   ii. Use series to evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x\arctan(2x)}$.

Solution:

(a) Let

$$\begin{aligned} u &= x & dv &= \sec^2 x\,dx \\ du &= dx & v &= \tan x \end{aligned}$$

and apply Integration by Parts.

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\underbrace{\sec^2 x\,dx}_{dv} = x\tan x - \int \tan x\,dx$$

$$= \boxed{x\tan x - \ln|\sec x| + C} \quad \text{OR} \quad \boxed{x\tan x + \ln|\cos x| + C}$$

Grading: No work is necessary for the integral of $\tan x$.

$$\int \tan x\,dx = \int \underbrace{\frac{\sin x}{\cos x}\,dx}_{\substack{u=\cos x\\ du=-\sin x\,dx}} = \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

(b) i. Use the binomial series:

$$\begin{aligned} (1+x)^{1/2} &\approx \binom{1/2}{0} + \binom{1/2}{1}x + \binom{1/2}{2}x^2 \\ &= 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 \\ &= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2} \end{aligned}$$

ii. The Maclaurin series for $x\arctan(2x)$ is

$$x\arctan(2x) = x\left(2x - \frac{(2x)^3}{3} + \cdots\right) = 2x^2 - \frac{8}{3}x^4 + \cdots$$

Insert this series and the binomial series for $\sqrt{1+x}$ into the limit.

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x\arctan(2x)} &= \lim_{x\to 0}\frac{\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\binom{1/2}{3}x^3+\cdots\right)-1-\frac{x}{2}}{2x^2-\frac{8}{3}x^4+\cdots} \\ &= \lim_{x\to 0}\frac{-\frac{1}{8}x^2+\binom{1/2}{3}x^3+\cdots}{2x^2-\frac{8}{3}x^4+\cdots}. \end{aligned}$$

Divide numerator and denominator by x^2, then substitute $x = 0$.

$$= \lim_{x\to 0} \frac{-\frac{1}{8} + \binom{1/2}{3}x + \cdots}{2 - \frac{8}{3}x^2 + \cdots}$$

$$= \frac{-\frac{1}{8} + 0}{2 - 0} = \boxed{-\frac{1}{16}}$$

4. (24 pts) Consider the region $\mathcal{R}$ bounded by the hyperbola $y^2 - \dfrac{x^2}{9} = 1$ and the lines $x = 0$, $x = 3$.

   (a) Sketch and shade the region $\mathcal{R}$. Label all intercepts.

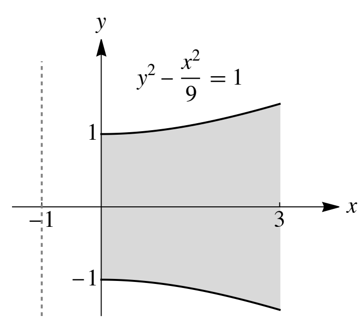
   (b) Set up integrals to find the following quantities. Simplify integrands but otherwise do not evaluate the integrals.

      i. Volume of the solid generated by rotating the region $\mathcal{R}$ about the y-axis

      ii. Area of the surface generated by rotating the upper half of the hyperbola, for $0 \le x \le 3$, about the line $x = -1$

Solution:

(a)



(b) The hyperbola corresponds to $y = \pm\sqrt{1 + \dfrac{x^2}{9}}$.

   i. Use the Shell Method. Let the radius $r = x$ and the height $h = 2\sqrt{1 + \dfrac{x^2}{9}}$. Then the volume is

$$V = \int_a^b 2\pi r h \, dx = \boxed{\int_0^3 4\pi x \sqrt{1 + \frac{x^2}{9}} dx}.$$

Alternate Solution
Use the Washer Method. Let $x = 3\sqrt{y^2-1}$. By symmetry, the volume $V = 2\left(V_1 + V_2\right)$, where V_1 is the volume of the solid between $y = 1$ and $y = \sqrt{2}$:

$$V_1 = \int_1^{\sqrt{2}} \pi\left(R^2 - r^2\right) dy = \int_1^{\sqrt{2}} \pi\left(3^2 - \left(3\sqrt{y^2-1}\right)^2\right) dy = \int_1^{\sqrt{2}} \pi\left(18 - 9y^2\right)\, dy$$

and V_2 is the volume of the circular cylinder between $y = 0$ and $y = 1$:

$$V_2 = \int_0^1 \pi\left(3^2\right)\, dy = \int_0^1 9\pi\, dy.$$

Therefore the volume V is

$$V = 2\left(V_1 + V_2\right) = 2\left(\int_1^{\sqrt{2}} \pi\left(18 - 9y^2\right)\, dy + \int_0^1 9\pi\, dy\right).$$

ii. The surface area equals $S = \displaystyle\int_a^b 2\pi r\, ds$ where $ds = \sqrt{1 + (y')^2}\, dx$. First calculate y'.

$$y = \sqrt{1 + \frac{x^2}{9}}$$

$$y' = \frac{1}{2}\left(1 + \frac{x^2}{9}\right)^{-1/2}\left(\frac{2x}{9}\right) = \frac{x}{9\sqrt{1+\frac{x^2}{9}}} \quad \text{OR} \quad \frac{x}{3\sqrt{9+x^2}} \quad \text{OR} \quad \frac{x}{\sqrt{81+9x^2}}.$$

Then

$$ds = \sqrt{1 + (y')^2}\, dx = \sqrt{1 + \frac{x^2}{81\left(1+\frac{x^2}{9}\right)}}\, dx = \sqrt{1 + \frac{x^2}{81+9x^2}}\, dx.$$

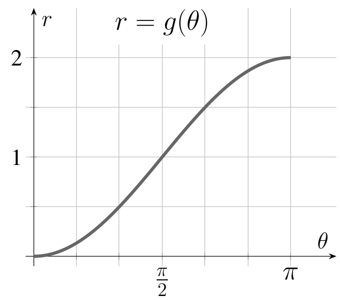
Let the radius $r = x + 1$. Then the surface area is

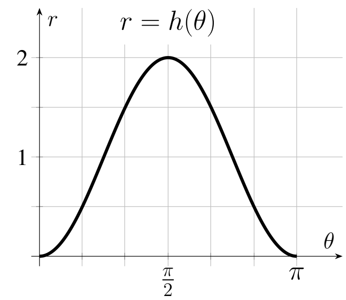
$$\begin{aligned} S = \int_a^b 2\pi r\, ds &= \boxed{\int_0^3 2\pi(x+1)\sqrt{1 + \frac{x^2}{81+9x^2}}\, dx} \quad \text{OR} \\ &= \int_0^3 2\pi(x+1)\sqrt{\frac{81+10x^2}{81+9x^2}}\, dx \quad \text{OR} \\ &= \int_0^3 \frac{2\pi}{3}(x+1)\sqrt{\frac{81+10x^2}{9+x^2}}\, dx. \end{aligned}$$

Alternate Solution
Let $x = 3\sqrt{y^2-1}$. Then $x' = \dfrac{3y}{\sqrt{y^2-1}}$ and

$$S = \boxed{\int_1^{\sqrt{2}} 2\pi\left(3\sqrt{y^2-1}+1\right)\sqrt{1 + \frac{9y^2}{y^2-1}}\, dy} = \int_1^{\sqrt{2}} 2\pi\left(3\sqrt{y^2-1}+1\right)\sqrt{\frac{10y^2-1}{y^2-1}}\, dy.$$

5. (34 pts)





(a) Sketch the polar curves $r = g(\theta)$ and $r = h(\theta)$, shown above as r-θ graphs, in the xy-plane. Label the curves and their x, y intercepts.

(b) Consider the intersection of the regions bounded by the polar curves $r = g(\theta)$ and $r = h(\theta)$ in quadrants I and II. The area of the intersection can be represented as the sum of two integrals. Set up, but do not evaluate, the integrals. Express your answer in terms of $g(\theta)$ and $h(\theta)$.

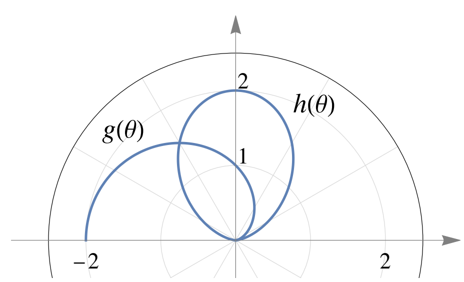
(c) The curve $r = g(\theta)$ can be represented in parametric form as

$$x = \cos\theta - \cos^2\theta, \quad y = \sin\theta - \tfrac{1}{2}\sin(2\theta).$$

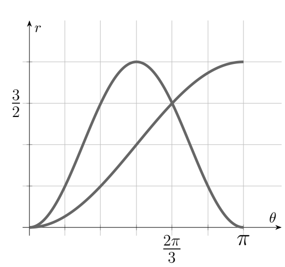
Find an equation for the line tangent to the polar curve at $\theta = \frac{\pi}{2}$.

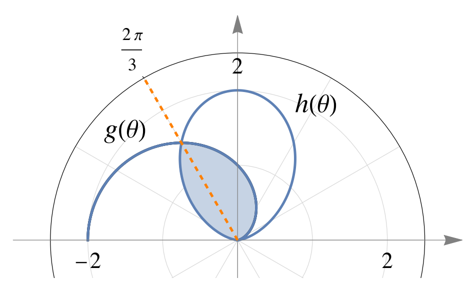
Solution:

(a)



(b) The curves intersect at the pole and at $\theta = \frac{2\pi}{3}$, $r = \frac{3}{2}$.





The ray $\theta = \frac{2\pi}{3}$ divides the intersection region into two subregions. The right subregion is bounded by $g(\theta)$ for $0 \leq \theta \leq \frac{2\pi}{3}$, and the left subregion is bounded by $h(\theta)$ for $\frac{2\pi}{3} \leq \theta \leq \pi$. Therefore the area of the intersection is

$$A = \boxed{\int_0^{2\pi/3} \frac{1}{2}\left(g(\theta)\right)^2\, d\theta + \int_{2\pi/3}^{\pi} \frac{1}{2}\left(h(\theta)\right)^2\, d\theta}.$$

(c) The derivative dy/dx equals

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left(\sin\theta - \frac{1}{2}\sin(2\theta)\right)}{\frac{d}{d\theta}\left(\cos\theta - \cos^2\theta\right)} = \frac{\cos\theta - \cos(2\theta)}{-\sin\theta + 2\cos\theta\sin\theta}.$$

At $\theta = \frac{\pi}{2}$ the tangent slope is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{0-(-1)}{-1+0} = -1.$$

The point of tangency is $(x, y) = (0, 1)$, so an equation for the tangent line is

$$\boxed{y = -x + 1}.$$