Work out the following problems and simplify your answers.

1. (30 pts) Evaluate the following integrals.

(a) $\displaystyle\int t\sin(3t)\,\mathrm{d}t$ (b) $\displaystyle\int_{-1}^{1}\frac{1}{x}\,\mathrm{d}x$

Solution:

(a) Using integration by parts, with $u = t$ and $\mathrm{d}v = \sin(3t)\,\mathrm{d}t$, we have

$$\int t\sin(3t)\,\mathrm{d}t = -\frac{1}{3}t\cos(3t) + \frac{1}{3}\int\cos(3t)\,\mathrm{d}t = \boxed{-\frac{1}{3}t\cos(3t) + \frac{1}{9}\sin(3t) + C.}$$

(b) There is an asymptote in $1/x$ at $x = 0$ meaning this integral is improper. With this in mind, we have

$$\int_{-1}^{1}\frac{1}{x}\,\mathrm{d}x = \int_{-1}^{0}\frac{1}{x}\,\mathrm{d}x + \int_{0}^{1}\frac{1}{x}\,\mathrm{d}x.$$

Evaluating the first integral, we have

$$\int_{-1}^{0}\frac{1}{x}\,\mathrm{d}x = \lim_{t\to 0^-}\int_{-1}^{t}\frac{1}{x}\,\mathrm{d}x = \lim_{t\to 0^-}\ln|x|\Big|_{-1}^{t} = \lim_{t\to 0^-}\ln|t| = -\infty$$

which diverges. Since the first integral diverges, the integral as a whole *diverges*.

2. (20 pts) Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{n+1}{n} - \dfrac{n+2}{n+1}$.

   (a) Does the sequence $\{a_n\}$ converge? If so, find its limit. If not, explain why not.

   (b) Using the sequence $\{a_n\}$ given in the problem, does $\sum_{n=1}^{\infty} a_n$ converge? If so, find its sum. If not, explain why not.

Solution:

(a) To test if the sequence converges, we will try to take its limit.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{1}{1} - \frac{1}{1} = \boxed{0.}$$

Since the limit is finite, the sequence *converges.*

(b) The sequence appears to telescoping so let's try that approach. The partial sums are given by

$$\begin{aligned} s_n = \sum_{i=1}^{n} \frac{i+1}{i} - \frac{i+2}{i+1} &= \left(\frac{2}{1} - \frac{3}{2}\right) + \left(\frac{3}{2} - \frac{4}{3}\right) + \left(\frac{4}{3} - \frac{5}{4}\right) + \cdots + \left(\frac{n+1}{n} - \frac{n+2}{n+1}\right) \\ &= \frac{2}{1} - \cancel{\frac{3}{2}} + \cancel{\frac{3}{2}} - \cancel{\frac{4}{3}} + \cancel{\frac{4}{3}} - \cancel{\frac{5}{4}} + \cdots + \cancel{\frac{n+1}{n}} - \frac{n+2}{n+1} \\ &= 2 - \frac{n+2}{n+1}. \end{aligned}$$

Taking the limit of our partial sum yields

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2 - \frac{n+2}{n+1} = 2 - 1 = 1.$$

Hence, the series *converges* and has a sum of $\boxed{1.}$

3. (20 pts) The Maclaurin series for $\operatorname{sinc} x = \dfrac{\sin x}{x}$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}$.

   (a) Find the radius of convergence of the series.

   (b) Using the series, what is the value of sinc(0)?

   (c) With the series above in mind, compute the sum of $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n+1)!\,6^{2n}}$.

Solution:

(a) To find the radius of convergence, we apply the ratio test to get

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+3)!}\,\frac{(2n+1)!}{(-1)^n x^{2n}}\right| = \left|\frac{x^2 x^{2n}}{(2n+3)(2n+2)(2n+1)!}\,\frac{(2n+1)!}{x^{2n}}\right| = \left|\frac{x^2}{(2n+3)(2n+2)}\right|$$

meaning

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0$$

implying the radius of convergence is $\boxed{R = \infty.}$

(b) Plugging in $x = 0$ into our series gives

$$\operatorname{sinc} 0 = \sum_{n=0}^{\infty}(-1)^n \frac{0^2 n}{(2n+1)!} = 1 + 0 + 0 + 0 + \cdots = \boxed{1.}$$

(c) To use our series for $\operatorname{sinc} x$, we first write

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n+1)!\,6^{2n}} = \sum_{n=0}^{\infty}(-1)^n \frac{(\pi/6)^{2n}}{(2n+1)!}.$$

Hence, the series is just the series for $\operatorname{sinc} x$ evaluated at $x = \pi/6$. So

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n+1)!\,6^{2n}} = \operatorname{sinc}\frac{\pi}{6} = \frac{\sin(\pi/6)}{\pi/6} = \boxed{\frac{3}{\pi}.}$$

4. (25 pts) The following problems are related.
   (a) Find the 3rd degree Taylor polynomial $T_3(x)$ centered at $a = 1$ of $\ln x$.
   (b) Estimate the error in using T_3 to approximate $\ln x$ at $x = \dfrac{3}{2}$.

Solution:

(a) To start, we will compute a table of the needed derivatives and function values as

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}$
0	$\ln x$	0	0
1	$1/x$	1	1
2	$-1/x^2$	-1	$-\frac{1}{2}$
3	$2/x^3$	2	$\frac{1}{3}$

Using the last column of the table, we have

$$\begin{aligned} T_3(x) &= 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\ &= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.} \end{aligned}$$

(b) To compute the error in $T_3(3/2)$, we need $f^{(4)}(z)$ which is given by

$$f^{(4)}(z) = -\frac{6}{z^4}.$$

Then using the Taylor Remainder Theorem, we know there exists some z between 1 and $3/2$ such that

$$|f(3/2) - T_3(3/2)| = \left|\frac{f^{(4)}(z)}{4!}(3/2-1)^4\right| = \left|\frac{-6/z^4}{24}(1/2)^4\right| = \frac{1}{64z^4}.$$

Since $1/z^4$ is decreasing, it is maximized at $z = 1$. Plugging this in gives our error estimate as

$$|f(3/2) - T_3(3/2)| = \frac{1}{64}z^4 \leq \boxed{\frac{1}{64}.}$$

5. (25 pts) Suppose the trajectory of a projectile launched from a cannon is given by the parametric curve

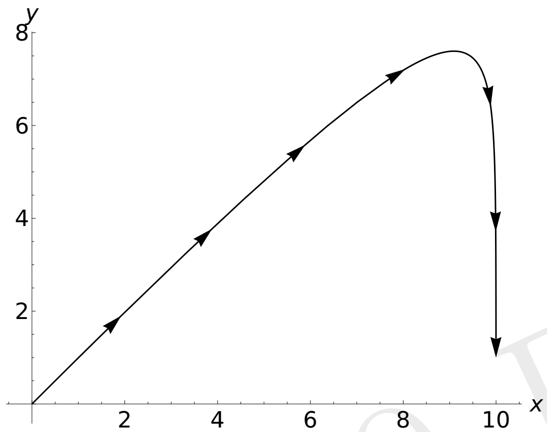
$$x = 10 - 10e^{-t},\ y = 11 - 11e^{-t} - t, \qquad t \geq 0$$

where t is the time from launch. Setup, **but do not evaluate**, integrals to find the following:

(a) The distance the projectile has traveled from $t = 0$ to $t = 10$.

(b) The area between the trajectory and the x-axis from $t = 1$ to $t = 5$.

Solution: It's not necessary, but if we plot the trajectory, we get



(a) To compute the distance traveled by the projectile, we just need to setup the arc length integral of the projectile. To start, we compute our needed derivatives as

$$x' = 10e^{-t}, \qquad y' = 11e^{-t} - 1.$$

Next, we can compute ds as

$$\mathrm{d}s = \sqrt{(x')^2 + (y')^2} = \sqrt{(10e^{-t})^2 + (11e^{-t} - 1)^2}\,\mathrm{d}t = \sqrt{221e^{-2t} - 22e^{-t} + 1}\,\mathrm{d}t.$$

Lastly, we can compute the distance as

$$L = \int_0^{10} \mathrm{d}s = \boxed{\int_0^{10} \sqrt{221e^{-2t} - 22e^{-t} + 1}\,\mathrm{d}t.}$$

(b) To find the area between the trajectory and the x-axis, we can use the parametric area formula

$$A = \int_a^b yx'\,\mathrm{d}t = \boxed{\int_1^5 (11 - 11e^{-t} - t)10e^{-t}\,\mathrm{d}t.}$$

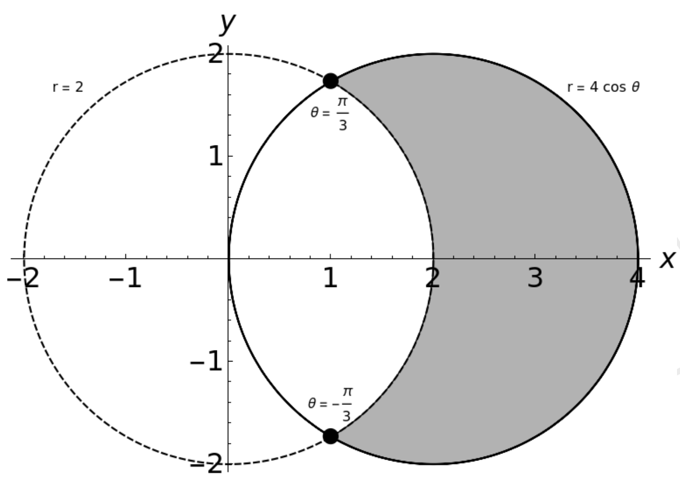
6. (30 pts) Consider the two polar equations $r = 4\cos\theta$ and $r = 2$. Answer the following:
   (a) Sketch both polar curves and label their intersections.
   (b) Find the area of the region inside of $r = 4\cos\theta$ and outside of $r = 2$.

Solution:

(a) $r = 4\cos\theta$ is given by the circle of diameter 4 opening to the right and $r = 2$ is just the circle of radius 2 centered at the origin. We can find the our intersection points by solving

$$4\cos\theta = 2 \implies \cos\theta = \frac{1}{2}$$

which gives $\theta = -\frac{\pi}{3}$ and $\theta = \frac{\pi}{3}$. Putting everything together, we get the plot



(b) From our graph in part (a), we can compute the area as

$$\begin{aligned}
A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4\cos\theta)^2 - \frac{1}{2}2^2 \, d\theta \\
&= \int_0^{\pi/3} (4\cos\theta)^2 - 2^2 \, d\theta \\
&= \int_0^{\pi/3} 16\cos^2\theta - 4 \, d\theta \\
&= \int_0^{\pi/3} 16\frac{1}{2}(1+\cos 2\theta) - 4 \, d\theta \\
&= \int_0^{\pi/3} 4 + 8\cos 2\theta \\
&= 4\theta + 4\sin 2\theta \Big|_0^{\pi/3} \\
&= \frac{4\pi}{3} + 4\sin\frac{2\pi}{3} \\
&= \boxed{\frac{4\pi}{3} + 2\sqrt{3}.}
\end{aligned}$$