Answer the following problems showing all of your work and simplifying your solutions.

1. (36 pts) Evaluate the following integrals. **Show all work!**

(a) $\displaystyle\int \frac{2x^4+4x^2+2}{x^3+2x}\,\mathrm{d}x$ (b) $\displaystyle\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x$ (c) $\displaystyle\int \sin^{-1}(x)\,\mathrm{d}x$

Solution:

(a) To start, the numerator is a higher degree polynomial than the denominator meaning we need long division.

$$\begin{array}{r|l} & 2x \\ \hline x^3+2x) & 2x^4+4x^2+2 \\ & -2x^4-4x^2 \\ \hline & 2 \end{array}$$

From long division, we have

$$\frac{2x^4+4x^2+2}{x^3+2x} = 2x + \frac{2}{x^3+2x} = 2x + \frac{2}{x(x^2+2)}.$$

Proceeding with partial fractions on the remainder, we have

$$\frac{2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2} \implies 2 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + Cx + 2A.$$

equating like-terms yields the system of equations

$$\begin{aligned} A+B &= 0 \\ C &= 0 \\ 2A &= 2 \end{aligned}$$

Solving the system yields $A=1, B=-1$, and $C=0$. Plugging in these coefficients and integrating yields

$$\int \frac{2x^4+4x^2+2}{x^3+2x}\,\mathrm{d}x = \int 2x + \frac{1}{x} - \frac{x}{x^2+2}\,\mathrm{d}x = \boxed{x^2 + \ln|x| - \frac{1}{2}\ln\left|x^2+2\right| + C.}$$

(b) The $\sqrt{1-x^2}$ prompts a trig sub with $x=\sin\theta$ which makes $\sqrt{1-x^2}=\cos\theta$ and $\mathrm{d}x=\cos\theta\,\mathrm{d}\theta$. Further, the new lower and upper bounds become $\theta=0$ and $\theta=\frac{\pi}{2}$. Plugging everything in yields

$$\begin{aligned} \int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x &= \int_0^{\pi/2} \sin^3\theta\cos^2\theta\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} \sin^2\theta\cos^2\theta\sin\theta\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} (1-\cos^2\theta)^2\cos^2\theta\sin\theta\,\mathrm{d}\theta \qquad u=\cos\theta,\ \mathrm{d}u=-\sin\theta\,\mathrm{d}\theta \\ &= -\int_1^0 (1-u^2)u^2\,\mathrm{d}u \\ &= \frac{1}{3} - \frac{1}{5} = \boxed{\frac{2}{15}.} \end{aligned}$$

(c) To integrate $\sin^{-1} x$ we need integration by parts with

$$u = \sin^{-1} x \implies \mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x, \quad \mathrm{d}v = \mathrm{d}x \implies v = x$$

giving

$$\begin{aligned} \int \sin^{-1} x\,\mathrm{d}x &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \qquad\qquad u = 1-x^2, \mathrm{d}u = -x\,\mathrm{d}x \\ &= x\sin^{-1} x + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= \boxed{x\sin^{-1} x + \sqrt{1-x^2} + C.} \end{aligned}$$

2. (24 pts) With $I = \int_0^4 \frac{1}{\sqrt{1+x}}\,\mathrm{d}x$, answer the following. **Leave your answer in exact form.**

   (a) Compute T_4 to estimate I. **Do not combine fractions.**

   (b) Find a reasonable bound for the error $|E_T|$ of your calculation in part (a).

   (c) What is the minimum number n of trapezoids required so that T_n is within 10^{-10} of the true solution I?

   (d) Suppose we change the bounds of integration to be $J = \int_{-1}^4 \frac{1}{\sqrt{1+x}}\,\mathrm{d}x$. Compute T_5 to approximate J. What happens to T_5 in this case?

Solution:

(a) With $n = 4, a = 0, b = 4$, we have $\Delta x = (4-0)/4 = 1$. Then,

$$T_4 = \frac{\Delta x}{2}\left(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right) = \boxed{\frac{1}{2}\left(\frac{1}{\sqrt{1}} + \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{4}} + \frac{1}{\sqrt{5}}\right).}$$

(b) To find an error bound, we need to find K. Well

$$f'(x) = -\frac{1}{2}\frac{1}{(1+x)^{3/2}} \implies |f''(x)| = \frac{3}{4}\left|\frac{1}{(1+x)^{5/2}}\right|.$$

Since $\frac{1}{(1+x)^{5/2}}$ is decreasing on 0 to 4, we have

$$|f''(x)| = \frac{3}{4}\left|\frac{1}{(1+x)^{5/2}}\right| \leq \frac{3}{4}\left|\frac{1}{(1+0)^{5/3}}\right| = \frac{3}{4} = K.$$

Plugging K into the error bound formula gives

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{\frac{3}{4}(4-0)^3}{12\cdot 4^2} = \boxed{\frac{1}{4}.}$$

(c) We require the error to be less than 10^{-10} which implies

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{\frac{3}{4}4^3}{12n^2} < 10^{-10}.$$

Rearranging the inequality for n yields

$$\boxed{n > \sqrt{\frac{\frac{3}{4}4^3}{12\cdot 10^{-10}}} = \sqrt{4\cdot 10^{10}} = 2\cdot 10^5}$$

meaning we need at least $n = 2\cdot 10^5 + 1$.

(d) With the new interval of integration, $a = -1, b = 4$, and $\Delta x = (4 - (-1))/5 = 1$. Plugging everything into the trapezoidal rule yields

$$\begin{aligned} T_5 &= \frac{\Delta x}{2}\left(f(-1) + 2f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right) \\ &= \frac{1}{2}\left(\underbrace{\frac{1}{0}}_{\infty} + \frac{2}{\sqrt{1}} + \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{4}} + \frac{1}{\sqrt{5}}\right) \\ &= \boxed{\infty \text{ (or DNE).}} \end{aligned}$$

From a direct calculation, we can see that T_5 breaks down (i.e. goes to infinity/DNE) at the asymptote on the left side of the integrand. Although the problem didn't ask to check, the improper integral J actually converges meaning the traditional trapezoidal rule is very poor at approximating improper integrals; some other approximation technique would be required here. Another note is that the value of K for the error bound is infinity with the new bounds of integration meaning the error is unbounded!

3. (20 pts) Do the following integrals converge or diverge? Fully justify your answers.

   (a) $\displaystyle\int_{-1}^{\infty} \frac{2}{(x-1)^3}\,\mathrm{d}x$

   (b) $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x^3}\,\mathrm{d}x$

Solution:

(a) Even though the integral is type-1 improper with an infinite length interval, the integral is also type-2 improper with an asymptote at $x = 1$. To evaluate, we need to break the integral into two chunks,

$$\int_{-1}^{1} \frac{2}{(x-1)^3}\,\mathrm{d}x + \int_{1}^{\infty} \frac{2}{(x-1)^3}\,\mathrm{d}x.$$

We will start by evaluating the first improper integral as

$$\begin{aligned}\int_{-1}^{1} \frac{2}{(x-1)^3}\,\mathrm{d}x &= \lim_{t\to 1^-} \int_{-1}^{t} \frac{2}{(x-1)^3}\,\mathrm{d}x \\ &= -\lim_{t\to 1^-} \left.\frac{1}{(x-1)^2}\right|_{-1}^{t} \\ &= -\lim_{t\to 1^-} \left(\frac{1}{(t-1)^2} - \frac{1}{(-2)^2}\right) \\ &= -\infty.\end{aligned}$$

Since one of the integrals diverges, the integral as a whole must also *diverge*.

(b) At a glance, the improper integral looks like a convergent p-integral with $p = 3$ or similar. To really show convergence, we should be able to find a larger convergent integral. Since $\ln x \leq x$ for all $x \geq 1$, we have

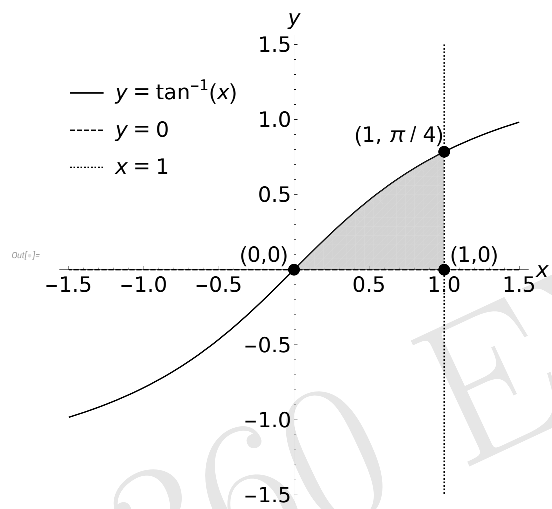
$$\frac{\ln x}{x^3} \leq \frac{x}{x^3} = \frac{1}{x^2}.$$

Further, $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ is a convergent p-integral with $p = 2$. Then by the Integral Comparison Test, the original integral must be *convergent*.

4. (20 pts) Let $\mathcal{R}$ be the region bounded by $y = \tan^{-1}(x)$, $y = 0$, and $x = 1$.
   (a) Sketch and shade the region $\mathcal{R}$. Label all axes, curves, and intersection points.
   (b) Set up, **but do not evaluate**, integrals to determine each of the following:
      i. The area of $\mathcal{R}$ using integration with respect to x.
      ii. The area of $\mathcal{R}$ using integration with respect to y.

Solution:

(a) Graphing and labeling everything leads to



(b) From the graph, we can setup area integrals as
   i. $A = \displaystyle\int_0^1 \tan^{-1} x \, \mathrm{d}x.$
   ii. $A = \displaystyle\int_0^{\pi/4} 1 - \tan y \, \mathrm{d}y.$

Trigonometric Identities

$$\cos^2(x) = \frac{1}{2}(1+\cos 2x) \quad \sin^2 x = \frac{1}{2}(1-\cos 2x) \quad \sin 2x = 2\sin x\cos x \quad \cos 2x = \cos^2 x - \sin^2 x$$

Inverse Trigonometric Integral Identities

$$\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, u^2 < a^2$$
$$\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C, u^2 > a^2$$

Midpoint Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)],\ \Delta x = \frac{b-a}{n},\ \overline{x}_i = \frac{x_{i-1}+x_i}{2},\ |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Trapezoidal Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)],\ \Delta x = \frac{b-a}{n},\ |E_T| \le \frac{K(b-a)^3}{12n^2}$$