Answer the following problems and simplify your answers.

1. (22 pts) Let $\mathcal{R}$ be the region bounded by the curves $y = x^2 - 2x$ and $y = 2x$.

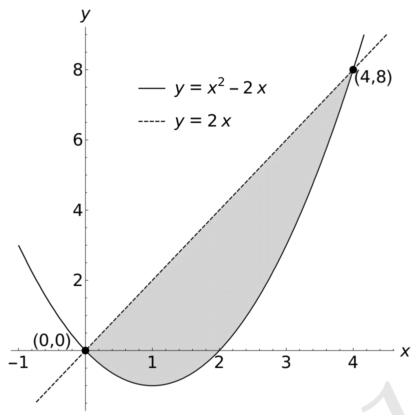
   (a) Sketch and shade the region $\mathcal{R}$, labeling the axes, intersection points, and curves.

   (b) Set up, **but do not evaluate**, an integral with respect to x to find the volume of the solid generated when $\mathcal{R}$ is revolved about the line $x = -1$.

   (c) Set up, **but do not evaluate**, an integral with respect to x to find the volume of the solid generated when $\mathcal{R}$ is revolved about the line $y = 8$.

Solution:

(a) Graphing everything, we have



(b) To integrate with respect to x while rotating around $x = -1$, we need cylindrical shells. In this case, $r = x + 1$ and $h = 2x - (x^2 - 2x)$. Then the volume can be computed via

$$\boxed{V = \int_0^4 2\pi(x+1)(2x - (x^2 - 2x))\,\mathrm{d}x.}$$

(c) To integrate with respect to x while rotating around $y = 8$, we need to use the washer method. In this case, $R = 8 - (x^2 - 2x)$ and $r = 8 - 2x$. Plugging in yields a volume integral of

$$\boxed{V = \pi \int_0^4 \left(8 - (x^2 - 2x)\right)^2 - (8 - 2x)^2\,\mathrm{d}x.}$$

2. (18 pts) Consider the curve $y = \ln(\cos(x))$ on the interval $0 \le x \le \pi/4$.

   (a) Set up, **but do not evaluate**, an integral to find the arc length of this curve. **Fully simplify** ds.

   (b) The curve is rotated about the x-axis to generate a surface. Set up, **but do not evaluate**, an integral to find the surface area.

Solution:

(a) To compute arc length, we need the derivative of y which requires the chain rule as

$$y' = -\frac{\sin x}{\cos x} = -\tan x.$$

Then, we the length element as

$$\mathrm{d}s = \sqrt{1 + (y')^2}\,\mathrm{d}x = \sqrt{1 + (-\tan x)^2}\,\mathrm{d}x = \sqrt{1 + \tan^2 x}\,\mathrm{d}x = \sec x\,\mathrm{d}x.$$

Integrating the length element yields

$$\boxed{L = \int_0^{\pi/4} \mathrm{d}s = \int_0^{\pi/4} \sec x\,\mathrm{d}x.}$$

(b) To find the surface area, we just need to find the radius r in terms of x since already have ds in terms of x. Since we are rotating about the x-axis, our radius is given by

$$r = y - 0 = \ln(\cos(x)).$$

Using the surface area formula, we have

$$\boxed{SA = \int_0^{\pi/4} 2\pi r\,\mathrm{d}s = \int_0^{\pi/4} 2\pi \ln(\cos(x))\sec(x)\,\mathrm{d}x.}$$

3. (24 pts) Assuming a uniform density ρ, find the centroid of the region under the curve $y = \cos(x)$ and above the x-axis with $0 \leq x \leq \pi/2$.

Solution: To start, we will compute the mass of the region as

$$m = \rho \int_0^{\pi/2} \cos x \, \mathrm{d}x = \rho \sin x \Big|_0^{\pi/2} = \rho.$$

Next, we can compute the moment about the y-axis using IBP with $u = x$ and $\mathrm{d}v = \cos(x)\, \mathrm{d}x$ to get

$$M_y = \rho \int_0^{\pi/2} x \cos x \, \mathrm{d}x = \rho \left(x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, \mathrm{d}x \right) = \rho \left(\frac{\pi}{2} + \cos x \Big|_0^{\pi/2} \right) = \rho \left(\frac{\pi}{2} - 1 \right).$$

Now, we just need to compute the moment about the x-axis as

$$M_x = \rho \int_0^{\pi/2} \frac{1}{2} \cos^2 x \, \mathrm{d}x = \rho \int_0^{\pi/2} \frac{1}{4} (1 + \cos(2x)) \, \mathrm{d}x = \rho \frac{1}{4} \left(x + \frac{1}{2} \sin(2x) \right) \Big|_0^{\pi/2} = \rho \frac{1}{8}.$$

Putting everything together, we have a centroid of

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \boxed{\left(\frac{\pi}{2} - 1, \frac{\pi}{8} \right).}$$

4. (12 pts) A 10 m long chain with a mass of 50 kg hangs vertically from a crane. How much work is required to lift the entire chain to the top? (Use 10 $\mathrm{m/s^2}$ as the acceleration due to gravity).

Solution: There are a few ways to approach work problems. Here will find the force function and then integrate over the length of the rope.

The force being applied is the weight of the rope under gravity. The linear density of the rope is given by

$$\rho = 50/10 = 5 \text{ kg/m}.$$

Then, the weight of the rope is given by $g \cdot m$ where m is the length of the rope. Assuming x is the length of rope we still need to pull up, we have a force function of

$$F(x) = g \cdot (50 - \rho x) = 10(50 - 5x) = 50(10 - x) \text{ N}.$$

Integrating this force over the length of the rope yields a work of

$$W = \int_0^{10} 50(10 - x)\,\mathrm{d}x = 50\left(10x - \frac{1}{2}x^2\right)\Big|_0^{10} = 50(100 - 50) = \boxed{2500 \text{ J.}}$$

5. (24 pts) Determine whether the sequences below converge or diverge. For the convergent sequences, find their limit.

(a) $a_n = \tan\left(\dfrac{\pi n^4 + 3n^2 - 1}{4n^4 - 5}\right)$ (b) $b_n = (-1)^n \dfrac{(n-1)!}{(n+1)!}$ (c) $c_n = \dfrac{(2/5)^n}{(1/4)^n}$

Solution:

(a) Using the continuity of limits, we can bring the limit inside to get

$$\lim_{n\to\infty} \tan\left(\frac{\pi n^4 + 3n^2 - 1}{4n^4 - 5}\right) = \tan\left(\lim_{n\to\infty} \frac{\pi n^4 + 3n^2 - 1}{4n^4 - 5}\right) = \tan\left(\lim_{n\to\infty} \frac{\pi + \frac{3}{n^2} - \frac{1}{n^4}}{4 - \frac{5}{n^4}}\right)$$
$$= \tan\left(\frac{\pi}{4}\right) = \boxed{1.}$$

(b) The alternating part of the sequence $\{b_n\}$ is a little troublesome so we will first consider

$$\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \frac{(n-1)!}{(n+1)!} = \lim_{n\to\infty} \frac{(n-1)!}{(n+1)(n)(n-1)!} = \lim_{n\to\infty} \frac{1}{(n+1)n} = 0.$$

Since $\lim\limits_{n\to\infty} |b_n| = 0$, we know $\{b_n\}$ converges with $\boxed{\lim\limits_{n\to\infty} b_n = 0.}$

(c) With a little algebra, we have

$$\lim_{n\to\infty} \frac{(2/5)^n}{(1/4)^n} = \lim_{n\to\infty} \left(\frac{8}{5}\right)^n = \infty$$

since $8/5 > 1$. As a result, $\{c_n\}$ diverges.

Trigonometric Identities

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \qquad \sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \sin 2x = 2\sin x\cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Inverse Trigonometric Integral Identities

$$\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, u^2 < a^2$$

$$\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C, u^2 > a^2$$

Center of Mass Formulas

$$M_y = \rho\int_a^b x(f(x)-g(x))\,\mathrm{d}x \qquad \bar{x} = \frac{M_y}{m}$$

$$M_x = \rho\int_a^b \frac{1}{2}\left((f(x))^2-(g(x))^2\right)\mathrm{d}x \qquad \bar{y} = \frac{M_x}{m}$$

$$m = \rho\int_a^b f(x)-g(x)\,\mathrm{d}x$$