Work out the following problems and simplify your answers.

1. (30 pts) Determine if the following series converge or diverge. Fully justify your answer and state which test you used.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ (c) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

Solution:

(a) With the n^2 in the denominator, we have a hunch that the series may converge. A possible route then would be to use a Direct Comparison with a bigger convergent series. Since $0 \leq \cos^2 n \leq n$ for all n, we have

$$\frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}.$$

Further, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$. By the Direct Comparison Test, the original series converges.

(b) There isn't an obvious comparison that we can make with ne^{-n}, but it does look like a integration by parts problem if the sum was an integral. To use the Integral test, we can note that $a_n = ne^{-n}$ is positive, continuous, and decreasing. Then

$$\int_1^{\infty} xe^{-x}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t xe^{-x} = \lim_{t\to\infty} -xe^{-x}\Big|_1^t + \int_1^t e^{-x}\,\mathrm{d}x = \lim_{t\to\infty} -te^{-t} + e^{-1} - e^{-t} + e^{-1} = 2e^{-1}$$

showing the integral converges. By the Integral Test, the original series converges.

(c) Applying the Root Test, we have

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5 = 2^5.$$

Since $L = 2^5 > 1$, the Root Test tells us that the original series diverges.

2. (15 pts) Suppose we have the series

$$s = \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3^2}{2\cdot 4}\right) + \ln\left(\frac{4^2}{3\cdot 5}\right) + \ln\left(\frac{5^2}{4\cdot 6}\right) + \cdots$$

(a) Find a simple expression for the partial sums s_n of the series s.

(b) Does the series converge or diverge? Fully justify your answer. If the series converges, find its sum.

Solution:

(a) To get a handle on the partial sum, let's look at the first few partial sums:

$$\begin{aligned}
s_1 &= \ln\left(\frac{2}{3}\right) \\
s_2 &= \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3^2}{2\cdot 4}\right) = \ln\left(\frac{2\cdot 3^2}{3\cdot 2\cdot 4}\right) = \ln\left(\frac{3}{4}\right) \\
s_3 &= \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3^2}{2\cdot 4}\right) + \ln\left(\frac{4^2}{3\cdot 5}\right) = \ln\left(\frac{2\cdot 3^2\cdot 4^2}{3\cdot 2\cdot 4\cdot 3\cdot 5}\right) = \ln\left(\frac{4}{5}\right) \\
s_4 &= \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3^2}{2\cdot 4}\right) + \ln\left(\frac{4^2}{3\cdot 5}\right) + \ln\left(\frac{5^2}{4\cdot 6}\right) = \ln\left(\frac{2\cdot 3^2\cdot 4^2\cdot 5^2}{3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4\cdot 6}\right) = \ln\left(\frac{5}{6}\right)
\end{aligned}$$

Following the pattern, we have that

$$\boxed{s_n = \ln\left(\frac{n+1}{n+2}\right).}$$

(b) With the partial sum in hand, we can compute the sum of the series as

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{n+2}\right) = \ln\left(\lim_{n\to\infty}\frac{n+1}{n+2}\right) = \ln(1) = \boxed{0}$$

meaning the series also converges.

3. (15 pts) Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{4}{(n!)^2}$.

   (a) Show that the series converges.

   (b) Estimate the error in using the partial sum s_3 to approximate s.

Solution:

(a) The series is alternating with $b_n = \frac{4}{(n!)^2}$. Now, clearly b_n is decreasing as the denominator grows with n. Further, $\lim_{n\to\infty} b_n = \frac{1}{\infty} = 0$. So by the Alternating Series Test, the series converges.

(b) The criteria for the Alternating Series Test were verified in part (a), so the Alternating Series Estimation Theorem applies. In other words, we know the error satisfies

$$|s - s_3| \leq b_{3+1} = b_4 = \frac{4}{(4!)^2} = \frac{4}{24^2} = \boxed{\frac{1}{144}}.$$

4. (25 pts) Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n!}(n-1)!$$

(a) Find the radius and interval of convergence for the power series $f(x)$

(b) Using interval notation, for what values of x is $f(x)$ absolutely convergent, conditionally convergent, and divergent?

Solution:

(a) Before we use any tests, we can simplify the factorials in the series as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n!}(n-1)! = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n(n-1)!}(n-1)! = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{4n}$$

Next, applying the Ratio Test yields

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{4(n+1)} \frac{4n}{(-1)^n (x-2)^n}\right| = \left|\frac{(x-2)^n (x-2)}{(n+1)} \frac{n}{(x-2)^n}\right| = \frac{n}{n+1}|x-2|$$

which gives a limit of

$$L = \lim_{n\to\infty} \frac{n}{n+1}|x-2| = |x-2|.$$

From the Ratio Test, we must have

$$L = |x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$$

for convergence (possibly including the endpoints). This inequality implies $\boxed{R = 1.}$ Further, we almost have the full interval of convergence but we need to check the endpoints. For the left endpoint when $x = 1$, we have

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-2)^n}{4n} = \sum_{n=0}^{\infty} \frac{1}{4n}$$

which is the divergent Harmonic Series multiplied by $1/4$. Moving to the right endpoint when $x = 3$, we have

$$f(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (3-2)^n}{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n}$$

which is the convergent Alternating Harmonic Series multiplied by $1/4$. Putting everything together, we find that the interval of convergence is $\boxed{(1, 2].}$

(b) From part (a), we know the series is

$$\boxed{\textit{divergent} \text{ for } (-\infty, 1] \cup (3, \infty).}$$

Further, the right endpoint gave the Alternating Harmonic Series which is conditionally convergent since the absolute value of the series yields the regular divergent Harmonic Series. As a result, $f(x)$ is

$$\boxed{\textit{conditionally convergent} \text{ for } \{3\}.}$$

Lastly, the Ratio Test gives absolute convergence so the from the $L < 1$ inequality, we have

$$\boxed{\textit{absolute convergence} \text{ for } (1, 3).}$$

5. (15 pts) Starting with the Maclaurin series for $\dfrac{1}{1-x}$, write out a power series for the function below and determine its radius of convergence without the use of the Ratio or Root Tests.

$$f(x) = \frac{5}{1-4x^2}.$$

Solution: We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$. Now to find our series, we just need to replace x with $4x^2$ in our geometric series and multiply by 5 to get

$$f(x) = \frac{5}{1-4x^2} = 5\sum_{n=0}^{\infty}(4x^2)^n = \boxed{\sum_{n=0}^{\infty} 5 \cdot 4^n x^{2n}.}$$

To find the radius of convergence we replace x with $4x^2$ in the original convergence criteria to get

$$\left|4x^2\right| < 1 \implies \left|x^2\right| < \frac{1}{4} \implies |x| \le \frac{1}{2}$$

meaning the radius of convergence is $\boxed{R = \frac{1}{2}.}$

Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$