- This exam is worth 100 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/092524 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

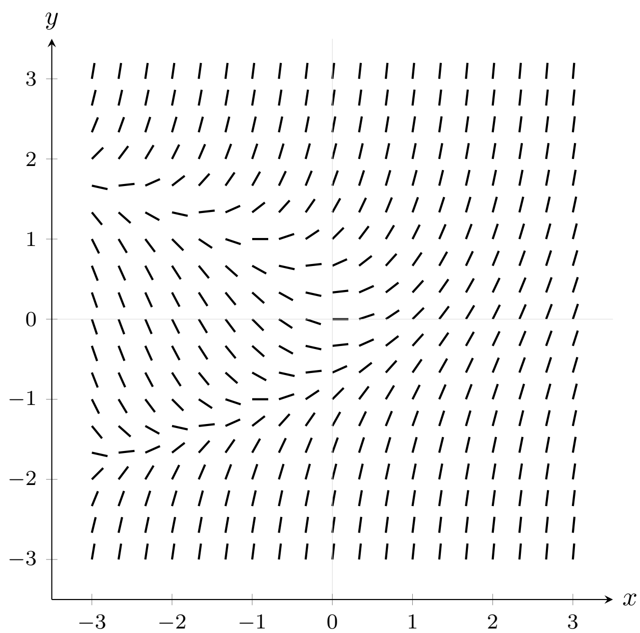
   (a) $\left(y^{(4)}\right)^2 + 2y' = 4$ is an eighth order differential equation.

   (b) The isoclines of $y' + 2y = 1 - t$ are lines with slope of -1.

   (c) Suppose L represents a linear operator and $L(\vec{\mathbf{u}}_1) = 3$ and $L(\vec{\mathbf{u}}_2) = 2$. Then $L(2\vec{\mathbf{u}}_1 - 3\vec{\mathbf{u}}_2) = 0$.

   (d) $y(x) = -\dfrac{1}{x^2}$ is a solution of the Bernoulli equation $xy' + y = x^2y^2$.

   (e) The following figure depicts the direction field of the differential equation $y' = x + y^2$.



2. [2360/092524 (12 pts)] Consider the initial value problem (IVP) $y' - \dfrac{4y}{t} = 2t^5e^{t^2},\ y(1) = 2e,\ t > 0$.

   (a) (10 pts) The first stage of the Euler-Lagrange Two Stage Method (variation of parameters) gives $y_h(t) = Ct^4$. Using this information, perform the second stage of the method to solve the IVP.

   (b) (2 pts) Identify the transient and steady state solutions, if any exist.

MORE PROBLEMS BELOW/ON REVERSE

3. [2360/092524 (16 pts)] Consider the differential equation $\frac{1}{4}\frac{\mathrm{d}y}{\mathrm{d}t} = 3 + 4t - y$.

   (a) (10 pts) Use the integrating factor method to find the general solution of the differential equation.

   (b) (6 pts) Estimate $y\left(\frac{1}{2}\right)$ using one step of Euler's method if $y(0) = 1$.

4. [2360/092524 (16 pts)] Consider the differential equation $(y-1)^3 y' + 4 = t$.

   (a) (6 pts) What does Picard's theorem tell us about the existence and unique of solutions to the initial value problem when

      i. $y(1) = 2$

      ii. $y(2) = 1$

   (b) (10 pts) Find the implicit solution to the equation that passes through the origin. The form of your final answer should have variables only on the left side, a number on the right side and contain no fractions.

5. [2360/092524 (17 pts)] Consider the differential equation $y' = (6-y)\,y - y^3$.

   (a) (6 pts) Find all equilibrium solutions.

   (b) (3 pts) Determine and state the stability of all equilibrium solutions.

   (c) (5 pts) Plot the phase line, graphically depicting the stability of the equilibrium solutions.

   (d) (3 pts) Using information from parts (a), (b), and (c), find $\lim_{t\to\infty} y(t)$ if

      i. $y(-1) = 0$

      ii. $y(-1) = 3$

      iii. $-4 < y(-1) < -1$

6. [2360/092524 (13 pts)] A 1000-gallon holding tank at a wastewater treatment plant initially contains 500 gallons of solution in which there are 100 pounds of decontaminating chemicals dissolved. To clean the tank, fresh water is pumped into the tank at a rate of 10 gallons per minute. Well-mixed wastewater is drained from the tank at the rate of 20 gallons per minute.

   (a) (10 pts) Write down, but **DO NOT SOLVE**, an initial value problem that models the amount of the decontaminating chemicals in the tank.

   (b) (3 pts) Will the solution (which you do not need to find) to the IVP in part (a) be valid for all $t \geq 0$? If not, give an appropriate interval over which the solution is valid. In either case, briefly justify your answer in words.

7. [2360/092524 (16 pts)] The following problems are not related.

   (a) (4 pts) In a certain competition model describing the populations of species P and Q, the point $(P, Q) = (0, 100)$ is the only equilibrium point (solution) and it is stable. What can you infer about the long term behavior of the population of each species from this information?

   (b) (12 pts) Consider the system

$$x' = y^3 - 8$$

$$y' = 4 - x^2$$

      i. (3 pts) Find all the h nullclines, if any.

      ii. (3 pts) Find all the v nullclines, if any.

      iii. (3 pts) Find all equilibrium solutions (points), if any exist.

      iv. (3 pts) In a phase plane, draw the vector at the origin, $(0, 0)$, indicating the direction of the trajectory there.