1. [2360/092524 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

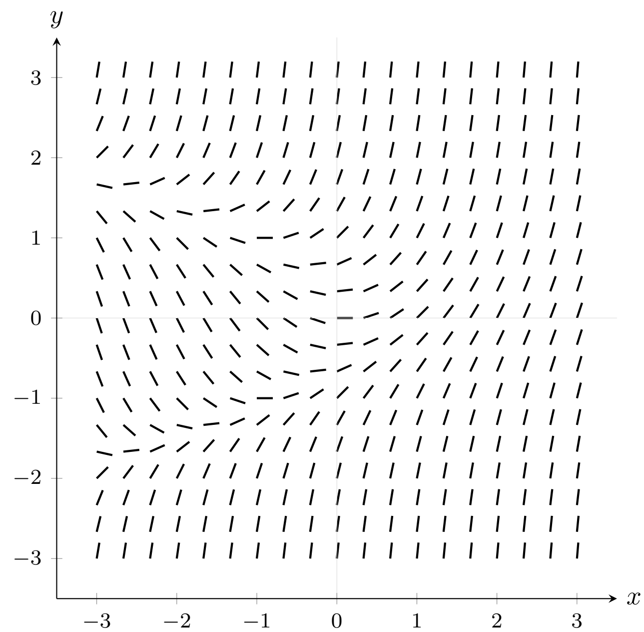
   (a) $\left(y^{(4)}\right)^2 + 2y' = 4$ is an eighth order differential equation.

   (b) The isoclines of $y' + 2y = 1 - t$ are lines with slope of -1.

   (c) Suppose L represents a linear operator and $L(\vec{\mathbf{u}}_1) = 3$ and $L(\vec{\mathbf{u}}_2) = 2$. Then $L(2\vec{\mathbf{u}}_1 - 3\vec{\mathbf{u}}_2) = 0$.

   (d) $y(x) = -\dfrac{1}{x^2}$ is a solution of the Bernoulli equation $xy' + y = x^2y^2$.

   (e) The following figure depicts the direction field of the differential equation $y' = x + y^2$.



SOLUTION:

(a) **FALSE** The equation is fourth order.

(b) **FALSE** $y' = -2y - t + 1 \implies f(t,y) = -2y - t + 1$. Isoclines are $-2y - t + 1 = k \implies y = -\dfrac{1}{2}t + \dfrac{1}{2} - \dfrac{k}{2}$ which are lines with slope $-1/2$.

(c) **TRUE** $L(2\vec{\mathbf{u}}_1 - 3\vec{\mathbf{u}}_2) = 2L(\vec{\mathbf{u}}_1) - 3L(\vec{\mathbf{u}}_2) = 2(3) - 3(2) = 0$

(d) **TRUE** $xy' + y = x\left(\dfrac{2}{x^3}\right) - \dfrac{1}{x^2} = \dfrac{2}{x^2} - \dfrac{1}{x^2} = \dfrac{1}{x^2} = x^2\left(-\dfrac{1}{x^2}\right)^2 = x^2y^2$

(e) **TRUE** Note, in particular, the slopes of the linear elements when $y = 0$ and $x = 0$.

2. [2360/092524 (12 pts)] Consider the initial value problem (IVP) $y' - \dfrac{4y}{t} = 2t^5e^{t^2},\ y(1) = 2e,\ t > 0$.

   (a) (10 pts) The first stage of the Euler-Lagrange Two Stage Method (variation of parameters) gives $y_h(t) = Ct^4$. Using this information, perform the second stage of the method to solve the IVP.

   (b) (2 pts) Identify the transient and steady state solutions, if any exist.

SOLUTION:

(a)

$$y_p = v(t)t^4$$

$$y_p' - 4\frac{y_p}{t} = 4t^3v(t) + t^4v'(t) - 4t^3v(t) = 2t^5e^{t^2}$$

$$v'(t) = 2te^{t^2}$$

$$v(t) = \int 2te^{t^2}\,\mathrm{d}t = e^{t^2}$$

$$y_p = t^4e^{t^2}$$

$$y = y_h + y_p = ct^4 + t^4e^{t^2} \quad \text{(apply the initial condition)}$$

$$y(1) = 2e = c + e \implies c = e$$

$$y(t) = t^4\left(e^{t^2} + e\right)$$

(b) The solution contains neither a transient nor a steady state solution.

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3. [2360/092524 (16 pts)] Consider the differential equation $\dfrac{1}{4}\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3 + 4t - y$.

(a) (10 pts) Use the integrating factor method to find the general solution of the differential equation.

(b) (6 pts) Estimate $y\left(\frac{1}{2}\right)$ using one step of Euler's method if $y(0) = 1$.

SOLUTION:

(a) Rewrite as $y' + 4y = 12 + 16t$. Then $p(t) = 4$ and the integrating factor is $\mu(t) = e^{4t}$.

$$\left(e^{4t}y\right)' = (12 + 16t)\,e^{4t}$$

$$e^{4t}y = \int (12 + 16t)\,e^{4t}\,\mathrm{d}t$$

$$\mathrm{d}v = e^{4t}\,\mathrm{d}t, u = 12 + 16t \implies v = \frac{1}{4}e^{4t}, \mathrm{d}u = 16\,\mathrm{d}t$$

$$e^{4t}y = \frac{1}{4}(12 + 16t)e^{4t} - 4\int e^{4t}\,\mathrm{d}t$$

$$e^{4t}y = 3e^{4t} + 4te^{4t} - e^{4t} + C = 2e^{4t} + 4te^{4t} + C$$

$$y(t) = 2 + 4t + Ce^{-4t}$$

(b) One step requires $h = \dfrac{1}{2}$. Euler's method is $y_{n+1} = y_n + \dfrac{1}{2}\left(12 + 16t_n - 4y_n\right), n = 0, 1, \ldots$

$$y\left(\frac{1}{2}\right) \approx y_1 = y_0 + \frac{1}{2}\left(12 + 16t_0 - 4y_0\right) = 1 + \frac{1}{2}\left[12 + 16(0) - 4(1)\right] = 1 + 4 = 5$$

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4. [2360/092524 (16 pts)] Consider the differential equation $(y-1)^3y' + 4 = t$.

(a) (6 pts) What does Picard's theorem tell us about the existence and unique of solutions to the initial value problem when

   i. $y(1) = 2$

   ii. $y(2) = 1$

(b) (10 pts) Find the implicit solution to the equation that passes through the origin. The form of your final answer should have variables only on the left side, a number on the right side and contain no fractions.

SOLUTION:

(a) Rewrite the differential equation as $y' = \dfrac{t-4}{(y-1)^3}$ so that $f(t,y) = \dfrac{t-4}{(y-1)^3}$ and $f_y(t,y) = \dfrac{-3(t-4)}{(y-1)^4}$. These two are defined for all t, y except $y = 1$.

  i. $f(1,2)$ and $f_y(1,2)$ are continuous in a rectangle containing $(1,2)$ (keep it small enough to not have $y = 1$ in it) so that Picard's theorem tells us that there is a unique solution to the IVP in some interval around $t = 1$.

  ii. $f(2,1)$ and $f_y(2,1)$ are not defined and thus there is no rectangle containing $(2,1)$ where $f(t,y)$ and $f_y(t,y)$ are continuous. Picard's theorem is inconclusive and tells us nothing about the existence or uniqueness of solutions to the IVP.

(b) Use separation of variables.

$$\int (y-1)^3 \, \mathrm{d}y = \int (t-4) \, \mathrm{d}t$$

$$\frac{(y-1)^4}{4} = \frac{t^2}{2} - 4t + C$$

$$(y-1)^4 = 2t^2 - 16t + C \qquad \text{apply the initial condition } \ y(0) = 0$$

$$(0-1)^4 = 2\left(0^2\right) - 16(0) + C \implies C = 1$$

$$(y-1)^4 - 2t^2 + 16t = 1$$

5. [2360/092524 (17 pts)] Consider the differential equation $y' = (6-y)\,y - y^3$.

(a) (6 pts) Find all equilibrium solutions.

(b) (3 pts) Determine and state the stability of all equilibrium solutions.

(c) (5 pts) Plot the phase line, graphically depicting the stability of the equilibrium solutions.

(d) (3 pts) Using information from parts (a), (b), and (c), find $\lim_{t\to\infty} y(t)$ if

  i. $y(-1) = 0$

  ii. $y(-1) = 3$

  iii. $-4 < y(-1) < -1$

SOLUTION:

(a)

$$f(y) = (6-y)\,y - y^3 = 6y - y^2 - y^3 = y(6 - y - y^2) = y(2-y)(3+y) = 0 \implies y = -3, 0, 2 \text{ are equlibrium solutions}$$

(b) $y < -3 \implies y' > 0; \quad -3 < y < 0 \implies y' < 0; \quad 0 < y < 2 \implies y' > 0; \quad 2 < y \implies y' < 0$

$y = -3$ is stable; $y = 0$ is unstable; $y = 2$ is stable

(c) Phase line.

$$\begin{array}{c} \downarrow \\ 2 \; \bullet \\ \uparrow \\ 0 \; \circ \\ \downarrow \\ -3 \; \bullet \\ \uparrow \end{array}$$

(d) i. 0

  ii. 2

  iii. -3

6. [2360/092524 (13 pts)] A 1000-gallon holding tank at a wastewater treatment plant initially contains 500 gallons of solution in which there are 100 pounds of decontaminating chemicals dissolved. To clean the tank, fresh water is pumped into the tank at a rate of 10 gallons per minute. Well-mixed wastewater is drained from the tank at the rate of 20 gallons per minute.

   (a) (10 pts) Write down, but **DO NOT SOLVE**, an initial value problem that models the amount of the decontaminating chemicals in the tank.

   (b) (3 pts) Will the solution (which you do not need to find) to the IVP in part (a) be valid for all $t \geq 0$? If not, give an appropriate interval over which the solution is valid. In either case, briefly justify your answer in words.

SOLUTION:

(a) Let $x(t)$ be the mass (lb) of decontaminating chemicals in the tank at time t. Since the inflow rate is less than the outflow rate, the tank will eventually empty. The initial value problem that governs the volume of solution in the tank is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow in } - \text{ flow out} = 10 - 20 = -10, V(0) = 500$$

$$V(t) = \int -10\,\mathrm{d}t = -10t + C; \quad V(0) = 500 = -10(0) + C \implies C = 500 \text{ and } V(t) = -10t + 500$$

For the amount of chemicals in the tank we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{mass rate in } - \text{ mass rate out} = \left(0\frac{\text{lb}}{\text{gal}}\right)\left(10\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x}{-10t+500}\frac{\text{lb}}{\text{gal}}\right)\left(20\frac{\text{gal}}{\text{min}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{-t+50} = 0, \; x(0) = 100$$

(b) The solution is valid on $0 \leq t < 50$. The tank is empty when $t = 50$ minutes.

7. [2360/092524 (16 pts)] The following problems are not related.

   (a) (4 pts) In a certain competition model describing the populations of species P and Q, the point $(P, Q) = (0, 100)$ is the only equilibrium point (solution) and it is stable. What can you infer about the long term behavior of the population of each species from this information?

   (b) (12 pts) Consider the system

$$x' = y^3 - 8$$
$$y' = 4 - x^2$$

      i. (3 pts) Find all the h nullclines, if any.

      ii. (3 pts) Find all the v nullclines, if any.

      iii. (3 pts) Find all equilibrium solutions (points), if any exist.

      iv. (3 pts) In a phase plane, draw the vector at the origin, $(0, 0)$, indicating the direction of the trajectory there.

SOLUTION:

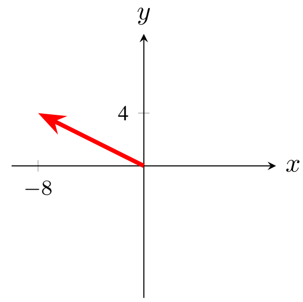
(a) Species P goes extinct and species Q approaches 100 individuals.

(b) i. h nullclines occur where $y' = 0$. This happens when $x = \pm 2$.

   ii. v nullclines occur where $x' = 0$. This happens when $y = 2$.

   iii. Equilibrium solutions (points) exist where both x' and y' vanish simultaneously, or where the h and v nullclines intersect. Thus, equilibrium solutions exist at $(2, 2)$ and $(-2, 2)$.

   iv.



y
4
-8
x

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