1. [2360/061424 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) Let L be a linear operator and suppose y_1 and y_2 are both solutions to the differential equation $L(\vec{\mathbf{y}}) = f(t)$. Then $y_1 - y_2$ is a solution of $L(\vec{\mathbf{y}}) = 0$.

   (b) Euler's method for the initial value problem $y' = 2,\ y(0) = 0$ will yield the exact solution for all t.

   (c) Picard's Theorem guarantees that the differential equation $y' = \sqrt{t+y},\ y(0) = 0$ does NOT have a unique solution.

   (d) Every first order linear homogeneous differential equation is separable.

   (e) If using the Euler-Lagrange Two Stage method (variation of parameters) to solve the differential equation $e^{-t}y' - y = e^{2-t}$, the particular solution has the form $y_p = v(t)e^{e^t}$.

SOLUTION:

(a) **TRUE** $L(y_1 - y_2) = L(y_1) - L(y_2) = f(t) - f(t) = 0$

(b) **TRUE** Euler's method is $y_{n+1} = y_n + 2h$. Thus $y_1 = 0 + 2h = 2h, y_2 = 2h + 2h = 4h, \dots$. These are points on the line $y = 2t$, the exact solution to the IVP.

(c) **FALSE** With $f(t,y) = \sqrt{t+y}$ we have $f_y(t,y) = \dfrac{1}{2\sqrt{t+y}}$ which is not defined at $(0,0)$. Thus Picard's Theorem tells us nothing about the uniqueness of solutions to the IVP.

(d) **TRUE** Linear first order homogeneous equations are of the form $a_1(t)y' + a_0(t)y = 0$ which can be rearranged to the form $y' = \left[\dfrac{a_0(t)}{a_1(t)}\right] y = f(t)g(y)$.

(e) **TRUE** The homogeneous problem is $y' - e^t y = 0$ which we solve as

$$\int \frac{\mathrm{d}y}{y} = \int e^t \,\mathrm{d}t$$
$$\ln|y| = e^t + k$$
$$|y| = e^k e^{e^t}$$
$$y = Ce^{e^t}$$

giving a particular solution of the form $y_p = v(t)e^{e^t}$.

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2. [2360/061424 (20 pts)] The following problems are not related.

   (a) (10 pts) You are making a secret marinade sauce for meat that involves dissolving 100 grams of Special Spice #1 in 10 gallons of vinegar in a large tank. A malefactor has decided to sabotage the mixture by creating a machine that pours 1 gallon, containing 30 grams of Special Spice #1, into the tank every minute. As soon as the machine is turned on, the malefactor also creates a hole in the tank that drains the well-mixed marinade from the tank at 2 gallons per minute. Set up, but **DO NOT SOLVE**, the initial value problem (IVP) modeling this scenario. Let $t = 0$ be the moment that the hole is created in the tank. Be sure to include the interval over which the equation is valid.

   (b) (10 pts) Use the integrating factor method to solve $(10-t)\dfrac{\mathrm{d}y}{\mathrm{d}t} + 2y = 300 - 30t,\ y(0) = 100$. Do not simply use formulas. Show all the steps needed to arrive at the solution.

SOLUTION:

(a) Let $x(t)$ be the amount of Special Spice #1 in the tank at time t and $V(t)$ the volume of solution in the tank t. Since the flows in and out differ, the volume of marinade in the tank will not be constant. Indeed

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow in } - \text{ flow out} = 1 - 2 = -1,\ V(0) = 10 \implies V(t) = 10 - t$$

The amount of spice in the tank is determined by rate of change of mass of spice = mass rate in of spice − mass rate out of spice

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(30\,\frac{\text{grams}}{\text{gallon}}\right)\left(1\,\frac{\text{gallon}}{\text{minute}}\right) - \left[\frac{x \text{ grams}}{(10-t) \text{ gallon}}\right]\left(2\,\frac{\text{gallon}}{\text{minute}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 30 - \frac{2x}{10-t},\ x(0) = 100$$

This is valid on the interval $0 \le t < 10$ since the tank is empty at 10 minutes.

(b) Rearrange the ODE to

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2y}{10-t} = 30$$

We find the integrating factor from

$$\int \frac{2}{10-t}\,\mathrm{d}t = -2\ln|10-t| = \ln|10-t|^{-2} \implies \mu(t) = e^{\ln|10-t|^{-2}} = (10-t)^{-2} \quad \text{since } t \in [0, 10)$$

Then

$$\left[(10-t)^{-2}y\right]' = 30(10-t)^{-2}$$

$$(10-t)^{-2}y = \int 30(10-t)^{-2}\,\mathrm{d}t = 30(10-t)^{-1} + C$$

$$y(t) = (10-t)^2\left[30(10-t)^{-1} + C\right]$$

Application of the initial condition gives

$$100 = (10-0)^2\left[30(10-0)^{-1} + C\right] \implies 1 = 3 + C \implies C = -2$$

so that the solution to the initial value problem is

$$y(t) = 30(10-t) - 2(10-t)^2 = (10-t)(30-20+2t) = (10-t)(10+2t) = 100 + 10t - 2t^2$$

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3. [2360/061424 (24 pts)] After discovering the culinary sabotage noted part (a) of the previous problem, you decide to make a new batch of marinade with your trademark 100 grams of Special Spice #1. In addition to being irresistibly delicious, Special Spice #1 is also an unstable radioactive material with a half-life of 10 days. Be sure to show all your work for all parts of this problem.

(a) (8 pts) Set up the corresponding initial value problem for this decay problem assuming that the new marinade is made at $t = 0$.

(b) (8 pts) How much Special Spice #1 will be left after t days?

(c) (8 pts) Your marinade will lose its flavor if under 10 grams of Special Spice #1 are left. How long after making a fresh batch of the marinade do have to use it?

SOLUTION:

(a) Let y be the amount of Special Spice #1. An exponential decay problem has the form $\mathrm{d}y/\mathrm{d}t = -ky, k > 0$ and initial condition $y(0) = y_0$. Since we are told the half-life is 10 days, $k = \ln 2/10$. Thus the initial value problem is

$$y' = -\frac{\ln 2}{10}y,\ y(0) = 100$$

(b) The solution of the DE is, by separation of variables,

$$\int \frac{\mathrm{d}y}{y} = -\int \frac{\ln 2}{10}\,\mathrm{d}t$$

$$\ln|y| = -\frac{\ln 2}{10}t + K$$

$$|y| = e^K e^{-(\ln 2/10)t}$$

$$y = Ce^{-(\ln 2/10)t} \qquad \text{apply initial condition}$$

$$100 = Ce^0 \implies C = 100$$

$$y(t) = 100e^{-(\ln 2/10)t}$$

(c) We need to determine when $y(t) = 10$.

$$\begin{aligned} 10 &= 100e^{(-\ln 2/10)t} \\ \ln\frac{1}{10} &= \ln e^{(-\ln 2/10)t} \\ -\ln 10 &= -\frac{\ln 2}{10}t \\ t &= \frac{10\ln 10}{\ln 2} \end{aligned}$$

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4. [2360/061424 (22 pts)] The following problems are not related.

(a) (16 pts) Consider the differential equation $y' = y^3 - 3y^2 - y + 3$.

i. (10 pts) Find all equilibrium solutions and their stability.

ii. (6 pts) Plot the phase line for the differential equation.

(b) (6 pts) Given the differential equation $y' + y = t^2$, draw the isoclines corresponding to slopes of $1, 0, -1$. Be sure to include the line segments showing the slope of the solutions on each isocline and label important points.

SOLUTION:

(a) We can write the ODE as $y' = y^2(y-3) - (y-3) = \left(y^2 - 1\right)(y-3) = (y+1)(y-1)(y-3)$ showing that the equilibrium solutions are $y = -1, y = 1, y = 3$. You can also use the Rational Roots Theorem to factor.

i.

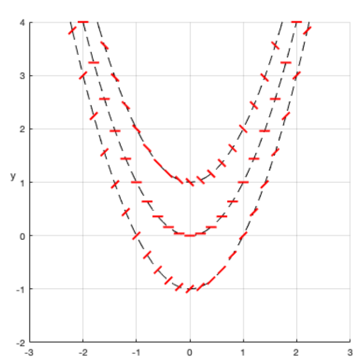
$$\begin{aligned} y > 3: &\quad y' > 0 \\ 1 < y < 3: &\quad y' < 0 \\ -1 < y < 1: &\quad y' > 0 \\ y < -1: &\quad y' < 0 \end{aligned}$$

Thus $y = -1$ and $y = 3$ are unstable and $y = 1$ is stable.

ii. Phase line.

$$\begin{array}{c} \uparrow \\ 3 \bigcirc \\ \downarrow \\ 1 \bullet \\ \uparrow \\ -1 \bigcirc \\ \downarrow \end{array}$$

(b) Isoclines are the parabolas $y = t^2 - k$ with $k = -1, 0, 1$.



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5. [2360/061424 (24 pts)] Consider the system of differential equations

$$x' = 2 - x + y$$
$$y' = y^2 - x$$

(a) (6 pts) Find the h nullclines, if any exist.

(b) (6 pts) Find the v nullclines, if any exist.

(c) (6 pts) Find the equilibrium solutions, if any exist.

(d) (6 pts) Draw a phase plane on your paper and put arrows in the phase plane showing the vector field at the points $(1, 1)$ and $(0, -2)$. Be sure to include appropriate labels.

SOLUTION:

(a) h nullclines are curves where $y' = 0$ and the line tangent to the trajectory is horizontal, so these are given by $x = y^2$.

(b) v nullclines are curves where $x' = 0$ and the line tangent to the trajectory is vertical, so these are given by $y = x - 2$.

(c) Equilibrium solutions occur where both $x' = 0$ and $y' = 0$, solutions of the system

$$x = y^2 \tag{1}$$
$$y = x - 2 \tag{2}$$

Substituting y from Eq. (2) into Eq. (1) yields $x = x^2 - 4x + 4 \implies x^2 - 5x + 4 = (x-4)(x-1) = 0 \implies x = 1, 4$ and from Eq. (2), $y = -1, 2$. The equilibrium solutions are thus $(1, -1)$ and $(4, 2)$. Note, that these can be found geometrically from the intersections of the h and v nullclines.

(d) Phase plane.