1. [2360/062824 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If $\mathbf{A}, \mathbf{B}$ are square matrices of the same order, then it is always the case that $\mathbf{AB} = \mathbf{BA}$

   (b) If $\mathbf{A}$ is a square matrix with $|\mathbf{A}| = 0$, then $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is inconsistent.

   (c) If $\mathbb{U}$ is the set of all points in the xy-plane except the origin, then $\mathbb{U}$ is not a vector space.

   (d) If $\mathbf{A}$ is an $n \times n$ matrix, then $\text{Tr}\,\mathbf{A} = \text{Tr}\left(\mathbf{A}^{\text{T}}\right)$.

   (e) If $|\mathbf{AB}| = 0$, and $\mathbf{A}$ is nonsingular, then 0 is an eigenvalue of $\mathbf{B}$.

   SOLUTION:

   (a) **FALSE** Whilst it is true that both $\mathbf{AB}$ and $\mathbf{BA}$ are defined for square matrices of the same order, the two products need not be equal. For example,

   $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix} \neq \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   (b) **FALSE** Homogeneous systems of linear equations always have at least the trivial solution and thus are consistent regardless of the value of the determinant of the coefficient matrix.

   (c) **TRUE** Vector spaces must contain the zero vector, which in the xy-plane $\left(\mathbb{R}^2\right)$ is the origin $(0, 0)$.

   (d) **TRUE** The trace of a matrix is the sum of the diagonal elements which do not change when a matrix is transposed.

   (e) **TRUE** If $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = 0$ and $\mathbf{A}$ is nonsingular, then $|\mathbf{A}| \neq 0$, implying that $|\mathbf{B}| = 0$, further implying that 0 is an eigenvalue of $\mathbf{B}$.

2. [2360/062824 (21 pts)] Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 0 \end{bmatrix}$.

   (a) (8 pts) Find the eigenvalues of $\mathbf{A}$ and give their algebraic multiplicities.

   (b) (13 pts) For each eigenvalue, find a basis for the eigenspace and give the geometric multiplicity of the eigenvalue.

   SOLUTION:

   (a) Since $\mathbf{A}$ is lower triangular, the eigenvalues are the diagonal elements. Thus, $\lambda = 1$ with algebraic multiplicity 1 and $\lambda = 0$ with algebraic multiplicity 2.

   (b) For $\lambda = 1$ we need to solve $(\mathbf{A} - 1\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

   $$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 6 & -3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 6 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \vec{\mathbf{v}} = \begin{bmatrix} 0 \\ -\frac{r}{3} \\ r \end{bmatrix}$$

   $$\implies \text{basis} \left\{ \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \text{ geometric multiplicity 1}$$

   For $\lambda = 0$ we need to solve $(\mathbf{A} - 0\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

   $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 6 & -3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \vec{\mathbf{v}} = \begin{bmatrix} \frac{r}{2} \\ r \\ s \end{bmatrix} \implies \text{basis} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ geometric multiplicity 2}$$

3. [2360/062824 (15 pts)] Which of the following subsets of $\mathbb{R}^3$ [parts (a) and (b)] and $\mathbb{M}_{22}$ [part (c)] are subspaces. Fully justify your answers.

(a) (5 pts) The solution set of $x_1 + x_2 + x_3 = 0$.

(b) (5 pts) Vectors in $\mathbb{R}^3$ having the form $\begin{bmatrix} a \\ a \\ 1 \end{bmatrix}$ where $a \in \mathbb{R}$.

(c) (5 pts) Matrices in $\mathbb{M}_{22}$ of the form $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ where $b \geq 0, a, c \in \mathbb{R}$.

SOLUTION:

(a) Subspace. This is a linear, homogeneous equation so the Superposition Principle applies. More specifically, solutions are of the form $\vec{\mathbf{x}} = \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix}, r, s \in \mathbb{R}$. Let $\vec{\mathbf{x}}_1 = \begin{bmatrix} -r_1 - s_1 \\ r_1 \\ s_1 \end{bmatrix}$ and $\vec{\mathbf{x}}_2 = \begin{bmatrix} -r_2 - s_2 \\ r_2 \\ s_2 \end{bmatrix}$ be in the solution set $\mathbb{W}$ and let $a, b \in \mathbb{R}$. Then

$$a\vec{\mathbf{x}}_1 + b\vec{\mathbf{x}}_2 = a\begin{bmatrix} -r_1 - s_1 \\ r_1 \\ s_1 \end{bmatrix} + b\begin{bmatrix} -r_2 - s_2 \\ r_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} -(ar_1 + br_2) - (as_1 + bs_2) \\ ar_1 + br_2 \\ as_1 + bs_1 \end{bmatrix} \in \mathbb{W}$$

(b) Not a subspace. The zero vector is not in the set. It is also not closed under either scalar multiplication or vector addition.

(c) Not a subspace. It is not closed under scalar multiplication. For example,

$$-1\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} -a & -b \\ b & -c \end{bmatrix} \notin \mathbb{W}$$

■

4. [2360/062824 (20 pts)] The following problems are not related. Provide full justification for your answers.

(a) (10 pts) Can the vectors $\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -12 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$?

(b) (10 pts) Let $\vec{\mathbf{p}}_1 = t^2 + 5$, $\vec{\mathbf{p}}_2 = 5t^2 + t$, $\vec{\mathbf{p}}_3 = 5t + 5$. Do constants c_1, c_2, c_3, not all zero, exist such that $c_1\vec{\mathbf{p}}_1 + c_2\vec{\mathbf{p}}_2 + c_3\vec{\mathbf{p}}_3 = 0$ for all real t?

SOLUTION:

(a) No, the vectors cannot form a basis for $\mathbb{R}^3$. The three vectors can only be a basis for the three dimensional vector space $\mathbb{R}^3$ if they are linearly independent. To check this, we need to see if the only solution to

$$c_1\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} -12 \\ 6 \\ -3 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is the trivial solution $c_1 = c_2 = c_3 = 0$. This is equivalent to finding the solutions of

$$\begin{bmatrix} 2 & -12 & -3 \\ 4 & 6 & 4 \\ -2 & -3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To this end,

$$\begin{vmatrix} 2 & -12 & -3 \\ 4 & 6 & 4 \\ -2 & -3 & -2 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} 6 & 4 \\ -3 & -2 \end{vmatrix} - 12(-1)^{1+2}\begin{vmatrix} 4 & 4 \\ -2 & -2 \end{vmatrix} - 3(-1)^{1+3}\begin{vmatrix} 4 & 6 \\ -2 & -3 \end{vmatrix} = 2(0) + 12(0) - 3(0) = 0$$

implying that the system has nontrivial solutions. Therefore, the vectors are linearly dependent and thus cannot form a basis for $\mathbb{R}^3$.

(b) No, constants c_1, c_2, c_3, not all zero, do not exist. We need to check for linear independence and we can try the Wronskian.

$$W(t) = \begin{vmatrix} t^2 + 5 & 5t^2 + t & 5t + 5 \\ 2t & 10t + 1 & 5 \\ 2 & 10 & 0 \end{vmatrix} = 2(-1)^{3+1}\begin{vmatrix} 5t^2 + t & 5t + 5 \\ 10t + 1 & 5 \end{vmatrix} + 10(-1)^{3+2}\begin{vmatrix} t^2 + 5 & 5t + 5 \\ 2t & 5 \end{vmatrix}$$

$$= 2\left[25t^2 + 5t - (50t^2 + 55t + 5)\right] - 10\left[5t^2 + 25 - (10t^2 + 10t)\right]$$

$$= 2\left(-25t^2 - 50t - 5\right) - 10\left(-5t^2 - 10t + 25\right)$$

$$= -260 \neq 0$$

implying that the functions are linearly independent so that only the constants $c_1 = c_2 = c_3 = 0$ will satisfy $c_1\vec{\mathbf{p}}_1 + c_2\vec{\mathbf{p}}_2 + c_3\vec{\mathbf{p}}_3 = 0$ for all real t.

■

5. [2360/062824 (16 pts)] The following problems are not related.

(a) (8 pts) Find all values of k such that $\mathbf{G}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has nontrivial solutions, where $\mathbf{G} = \begin{bmatrix} 1 & 0 & k \\ 0 & k & 0 \\ k & 0 & 9 \end{bmatrix}$.

(b) (8 pts) If $\mathbf{B}$ is an invertible matrix with $\mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, find the solution of $\mathbf{B}^{\mathrm{T}}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$

SOLUTION:

(a) We need to find values of k that make $|\mathbf{G}| = 0$.

$$\begin{vmatrix} 1 & 0 & k \\ 0 & k & 0 \\ k & 0 & 9 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} k & 0 \\ 0 & 9 \end{vmatrix} + k(-1)^{1+3}\begin{vmatrix} 0 & k \\ k & 0 \end{vmatrix} = 9k - k^3 = k(3-k)(3+k) = 0 \implies k = -3, 0, 3$$

(b) Since $\mathbf{B}$ is invertible, so is $\mathbf{B}^{\mathrm{T}}$. Thus

$$\vec{\mathbf{x}} = \left(\mathbf{B}^{\mathrm{T}}\right)^{-1}\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = \left(\mathbf{B}^{-1}\right)^{\mathrm{T}}\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \\ 10 \end{bmatrix}$$

■

6. [2360/062824 (18 pts)] Some friends of yours need the solution to a linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. They have already performed a number of elementary row operations on the augmented matrix, and have given you the following:

$$\left[\begin{array}{cccc|c} 0 & -2 & 0 & -6 & -10 \\ 0 & 1 & 2 & 7 & 11 \\ 0 & 1 & 0 & 3 & 5 \end{array}\right]$$

Your friends (and grader) have several requests:

(a) (6 pts) Find a particular solution of the system.

(b) (8 pts) Find a basis for the solution space of the associated homogeneous system. What is its dimension?

(c) (4 pts) Find the (general) solution, $\vec{\mathbf{x}}$, to the problem.

SOLUTION:

(a)

$$\left[\begin{array}{cccc|c} 0 & -2 & 0 & -6 & -10 \\ 0 & 1 & 2 & 7 & 11 \\ 0 & 1 & 0 & 3 & 5 \end{array}\right] \xrightarrow[R_2^*=-1R_3+R_2]{R_1^*=2R_3+R_1} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 1 & 0 & 3 & 5 \end{array}\right] \xrightarrow[R_2^*=\frac{1}{2}R_2]{R_1\leftrightarrow R_3} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} x_2 = 5 - 3x_4 \\ x_3 = 3 - 2x_4 \\ x_1, x_4 \text{ free} \end{array} \implies \vec{\mathbf{x}} = \begin{bmatrix} r \\ 5-3s \\ 3-2s \\ s \end{bmatrix} \implies \vec{\mathbf{x}}_p = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

(b)

$$\vec{\mathbf{x}}_h = \begin{bmatrix} r \\ -3s \\ -2s \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -3 \\ -2 \\ 1 \end{bmatrix}, \; r, s \in \mathbb{R} \implies \text{basis} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ dimension 2}$$

(c)

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \; r, s \in \mathbb{R}$$

■