- This exam is worth 100 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/071224 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

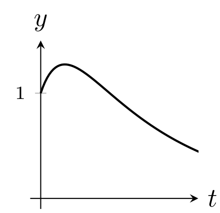
   (a) If $s > 0$, then $\int_0^\infty t^5 e^{-st}\,\mathrm{d}t = \frac{5}{s^5}$.

   (b) The solutions to the harmonic oscillator equation $2\ddot{x} + 7p\dot{x} + 6x = 12\cos\sqrt{3}t$ will grow without bound for only a single real value of p.

   (c) The differential equation $y'' + y\cos y = -5$ describes a conservative system.

   (d) $\mathscr{L}^{-1}\left\{\frac{3}{s^2+6s+12}\right\} = e^{-3t}\sin\sqrt{3}t$

   (e) The following graph depicts the solution of $y'' + 6y' + 5y = 0,\ y(0) = 1,\ y'(0) = 0$.



2. [2360/071224 (20 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = y''' - 6y'' + 13y' - 10y$.

   (a) (12 pts) If $y(t) = e^{2t}$ is a solution of $L(\vec{\mathbf{y}}) = 0$, find the general solution of $L(\vec{\mathbf{y}}) = 0$.

   (b) (8 pts) Convert the initial value problem $L(\vec{\mathbf{y}}) = e^{-t} + 5,\ y(0) = y'(0) = y''(0) = 0$, into a system of first order initial value problems. If possible, write the system using matrices and vectors. If not, explain why not.

3. [2360/071224 (30 pts)] A 1-kg mass is attached to a spring. The system is hooked up to a mechanism that imparts a damping force equal to eight times the instantaneous velocity of the mass. The entire apparatus is lying horizontally on a table.

   (a) (3 pts) You have been told that a force of 10 N is required to stretch the spring 0.5 m. Find the restoring (spring) constant and write the differential equation that governs the motion of the mass, assuming the oscillator is unforced.

   (b) (5 pts) Verify that the roots of the characteristic equation associated with the differential equation in part (a) are $-4 \pm 2i$ and write the equation of motion for the unforced oscillator, that is, the solution to the differential equation in part (a).

   (c) (3 pts) Is the oscillator overdamped, underdamped or critically damped? Justify your answer.

   (d) (15 pts) Now suppose the oscillator is subjected to an external force of $f(t) = 32\sin 2t + 16\cos 2t$. If the mass is initially $(t = 0)$ at rest at the equilibrium position, where is the mass when $t = \pi/4$?

   (e) (4 pts) If they exist, identify the transient and steady state solutions in part (d). If one or both do not exist, explain why not.

4. [2360/071224 (15 pts)] Use variation of parameters to find the general solution of $2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4\frac{\mathrm{d}y}{\mathrm{d}t} - 6y = 72te^{2t}$. Feel free to use the following formula if desired:

$$\int te^{at}\,\mathrm{d}t = \frac{(at-1)e^{at}}{a^2} + C$$

5. [2360/071224 (25 pts)] The following problems require the use of the Method of Undetermined Coefficients. **Do not solve** for the coefficients in parts (a) and (b).

   (a) (5 pts) Find the form of the particular solution when solving $w'' + 4w' - 5w = 2360 + 2\sin 3t + 3\cos 2t$.

   (b) (5 pts) Find the form of the particular solution if the characteristic equation is $r^4 - r^3 = 0$ and the forcing function (right hand side of the differential equation) is $f(t) = t^3 - 7e^{-t} + 2$.

   (c) (15 pts) Solve the initial value problem $z'' - 6z' + 9z = 6e^{3t},\ z(0) = 3,\ z'(0) = 12$.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
