1. [2360/071224 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

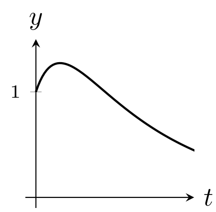
   (a) If $s > 0$, then $\displaystyle\int_0^\infty t^5 e^{-st}\,\mathrm{d}t = \frac{5}{s^5}$.

   (b) The solutions to the harmonic oscillator equation $2\ddot{x} + 7p\dot{x} + 6x = 12\cos\sqrt{3}t$ will grow without bound for only a single real value of p.

   (c) The differential equation $y'' + y\cos y = -5$ describes a conservative system.

   (d) $\mathscr{L}^{-1}\left\{\dfrac{3}{s^2+6s+12}\right\} = e^{-3t}\sin\sqrt{3}t$

   (e) The following graph depicts the solution of $y'' + 6y' + 5y = 0,\ y(0) = 1,\ y'(0) = 0$.



SOLUTION:

(a) **FALSE** $\displaystyle\int_0^\infty t^5 e^{-st}\,\mathrm{d}t = \mathscr{L}\left\{t^5\right\} = \frac{5!}{s^6}$

(b) **TRUE** Since $\omega_0 = \sqrt{6/2} = \sqrt{3} = \omega_f$, the system can be in resonance if it is undamped. This will occur if and only if $p = 0$.

(c) **TRUE** It is in the form $\ddot{y} + V'(y) = 0$ with $V'(y) = y\cos y + 5$.

(d) **FALSE** $\mathscr{L}^{-1}\left\{\dfrac{3}{s^2+6s+12}\right\} = \sqrt{3}\mathscr{L}^{-1}\left\{\dfrac{\sqrt{3}}{(s+3)^2+\left(\sqrt{3}\right)^2}\right\} = \sqrt{3}e^{-3t}\sin\sqrt{3}t$

(e) **FALSE** The graph has a positive slope at $t = 0$ but the initial condition is $y'(0) = 0$.

2. [2360/071224 (20 pts)] Consider the linear operator $L\left(\vec{\mathbf{y}}\right) = y''' - 6y'' + 13y' - 10y$.

   (a) (12 pts) If $y(t) = e^{2t}$ is a solution of $L\left(\vec{\mathbf{y}}\right) = 0$, find the general solution of $L\left(\vec{\mathbf{y}}\right) = 0$.

   (b) (8 pts) Convert the initial value problem $L\left(\vec{\mathbf{y}}\right) = e^{-t} + 5,\ y(0) = y'(0) = y''(0) = 0$, into a system of first order initial value problems. If possible, write the system using matrices and vectors. If not, explain why not.

SOLUTION:

(a) Since e^{2t} is a solution to the differential equation, $r = 2$ is solution to the characteristic equation $r^3 - 6r^2 + 13r - 10 = 0$. Synthetic division then gives

$$\begin{array}{c|rrrr} & 1 & -6 & 13 & -10 \\ 2 & & 2 & -8 & 10 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

which gives the characteristic equation as $(r-2)(r^2-4r+5) = 0$. Using the quadratic formula shows that the other two roots are $r = 2 \pm i$. From this the general solution is $y(t) = e^{2t}\left(c_1 + c_2\cos t + c_3\sin t\right)$.

(b) Let $x_1 = y, x_2 = y', x_3 = y''$. Then

$$x_1' = y' = x_2$$

$$x_2' = y'' = x_3$$

$$x_3' = y''' = e^{-t} + 5 + 10y - 13y' + 6y'' = 10x_1 - 13x_2 + 6x_3 + e^{-t} + 5$$

and since the equation is linear

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -13 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{-t}+5 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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3. [2360/071224 (30 pts)] A 1-kg mass is attached to a spring. The system is hooked up to a mechanism that imparts a damping force equal to eight times the instantaneous velocity of the mass. The entire apparatus is lying horizontally on a table.

   (a) (3 pts) You have been told that a force of 10 N is required to stretch the spring 0.5 m. Find the restoring (spring) constant and write the differential equation that governs the motion of the mass, assuming the oscillator is unforced.

   (b) (5 pts) Verify that the roots of the characteristic equation associated with the differential equation in part (a) are $-4 \pm 2i$ and write the equation of motion for the unforced oscillator, that is, the solution to the differential equation in part (a).

   (c) (3 pts) Is the oscillator overdamped, underdamped or critically damped? Justify your answer.

   (d) (15 pts) Now suppose the oscillator is subjected to an external force of $f(t) = 32\sin 2t + 16\cos 2t$. If the mass is initially $(t = 0)$ at rest at the equilibrium position, where is the mass when $t = \pi/4$?

   (e) (4 pts) If they exist, identify the transient and steady state solutions in part (d). If one or both do not exist, explain why not.

SOLUTION:

(a) If 10 N are required to stretch the spring 0.5 m, the restoring constant is $10/0.5 = 20$ N/m. If $x(t)$ is the displacement of the mass from its equilibrium position, the differential equation governing the motion is $\ddot{x} + 8\dot{x} + 20x = 0$.

(b) The characteristic equation is $r^2 + 8r + 20 = 0 \implies r = \dfrac{-8 \pm \sqrt{8^2 - 4(1)(20)}}{2} = \dfrac{-8 \pm 4i}{2} = -4 \pm 2i$ so that the equation of motion is $x(t) = e^{-4t}\left(c_1 \cos 2t + c_2 \sin 2t\right)$.

(c) Since $b^2 - 4mk = -16 < 0$, the oscillator is underdamped.

(d) From part (b) we have $x_h(t) = e^{-4t}\left(c_1 \cos 2t + c_2 \sin 2t\right)$. We use the Method of Undetermined Coefficients to find the particular solution.

$$x_p = A\cos 2t + B\sin 2t$$

$$\ddot{x}_p + 8\dot{x}_p + 20x_p = -4A\cos 2t - 4B\sin 2t + 8\left(-2A\sin 2t + 2B\cos 2t\right) + 20\left(A\cos 2t + B\sin 2t\right) = 32\sin 2t + 16\cos 2t$$

$$(16A + 16B)\cos 2t + (16B - 16A)\sin 2t = 32\sin 2t + 16\cos 2t$$

$$\left.\begin{aligned} A + B &= 1 \\ -A + B &= 2 \end{aligned}\right\} \implies B = \frac{3}{2}, A = -\frac{1}{2}$$

$$x_p = \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t$$

$$x(t) = e^{-4t}\left(c_1\cos 2t + c_2\sin 2t\right) + \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t \qquad \text{apply initial conditions}$$

$$x(0) = c_1 - \frac{1}{2} = 0 \implies c_1 = \frac{1}{2}$$

$$x'(t) = e^{-4t}\left(-\sin 2t + 2c_2\cos 2t\right) - 4e^{-4t}\left(\frac{1}{2}\cos 2t + c_2\sin 2t\right) + 3\cos 2t + \sin 2t$$

$$x'(0) = 2c_2 - 2 + 3 = 0 \implies c_2 = -\frac{1}{2}$$

$$x(t) = \frac{1}{2}e^{-4t}\left(\cos 2t - \sin 2t\right) + \frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t$$

$$x(\pi/4) = \frac{1}{2}e^{-4(\pi/4)}\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) + \frac{3}{2}\sin\frac{\pi}{2} - \frac{1}{2}\cos\frac{\pi}{2} = \frac{1}{2}\left(3 - e^{-\pi}\right)$$

(e) Transient solution: $\frac{1}{2}e^{-4t}\left(\cos 2t - \sin 2t\right)$; steady state solution: $\frac{3}{2}\sin 2t - \frac{1}{2}\cos 2t$

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4. [2360/071224 (15 pts)] Use variation of parameters to find the general solution of $2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4\frac{\mathrm{d}y}{\mathrm{d}t} - 6y = 72te^{2t}$. Feel free to use the following formula if desired:

$$\int te^{at}\,\mathrm{d}t = \frac{(at-1)e^{at}}{a^2} + C$$

SOLUTION:

Begin by getting the equation into the standard form: $y'' - 2y' - 3y = 36te^{2t}$. The characteristic equation for the associated homogeneous equation is $r^2 - 2r - 3 = (r-3)(r+1) = 0 \implies r = -1, 3 \implies y_1 = e^{-t}, y_2 = e^{3t}$. Let $y_p = v_1y_1 + v_2y_2$.

$$W\left[e^{-t}, e^{3t}\right] = \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = 4e^{2t}$$

$$v_1 = \int \frac{-y_2 f}{W\left[e^{-t}, e^{3t}\right]}\,\mathrm{d}t = \int \frac{-e^{3t}\left(36te^{2t}\right)}{4e^{2t}}\,\mathrm{d}t = -9\int te^{3t}\,\mathrm{d}t = -9\left[\frac{1}{9}(3t-1)e^{3t}\right] = e^{3t}(1-3t)$$

$$v_2 = \int \frac{y_1 f}{W\left[e^{-t}, e^{3t}\right]}\,\mathrm{d}t = \int \frac{e^{-t}\left(36te^{2t}\right)}{4e^{2t}}\,\mathrm{d}t = 9\int te^{-t}\,\mathrm{d}t = -9e^{-t}(t+1)$$

$$y_p = e^{3t}(1-3t)e^{-t} - 9e^{-t}(t+1)e^{3t} = e^{2t}(1-3t-9t-9) = -4e^{2t}(3t+2)$$

$$y = c_1e^{-t} + c_2e^{3t} - 4e^{2t}(3t+2)$$

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5. [2360/071224 (25 pts)] The following problems require the use of the Method of Undetermined Coefficients. **Do not solve** for the coefficients in parts (a) and (b).

   (a) (5 pts) Find the form of the particular solution when solving $w'' + 4w' - 5w = 2360 + 2\sin 3t + 3\cos 2t$.

   (b) (5 pts) Find the form of the particular solution if the characteristic equation is $r^4 - r^3 = 0$ and the forcing function (right hand side of the differential equation) is $f(t) = t^3 - 7e^{-t} + 2$.

   (c) (15 pts) Solve the initial value problem $z'' - 6z' + 9z = 6e^{3t},\ z(0) = 3,\ z'(0) = 12$.

SOLUTION:

(a) $r^2 + 4r - 5 = (r+5)(r-1) = 0 \implies r = -5, 1 \implies$ basis for the solution space of homogeneous problem is $\left\{e^{-5t}, e^t\right\}$. Thus, $y_p = A\sin 3t + B\cos 3t + C\sin 2t + D\cos 2t + E$

(b) $r^3(r-1) = 0 \implies r = 0,$ multiplicity 3 and $r = 1$ so a basis for the solution space is $\left\{1, t, t^2, e^t\right\}$. Thus, $y_p = t^3\left(At^3 + Bt^2 + Ct + D\right) + Ee^{-t} = At^6 + Bt^5 + Ct^4 + Dt^3 + Ee^{-t}$

(c) $r^2 - 6r + 9 = (r-3)^2 = 0 \implies r = 3$ multiplicity 2 $\implies z_h = c_1e^{3t} + c_2te^{3t}$. The initial guess for the particular solution is Ae^{3t} but this must be multiplied by t^2 so that none of the guess is a solution to the homogeneous problem. Thus $z_p = At^2e^{3t}$ and

$$z_p'' - 6z_p' + 9z_p = A\left(9t^2e^{3t} + 12te^{3t} + 2e^{3t}\right) - 6A\left(3t^2e^{3t} + 2te^{3t}\right) + 9At^2e^{3t} = 6e^{3t}$$

$$2Ae^{3t} = 6e^{3t} \implies A = 3 \implies z_p = 3t^2e^{3t}$$

$$z(t) = z_h(t) + z_p(t) = c_1e^{3t} + c_2te^{3t} + 3t^2e^{3t} \quad \text{apply initial conditions}$$

$$z(0) = c_1 = 3$$

$$z(t) = 3e^{3t} + c_2te^{3t} + 3t^2e^{3t} \implies z'(t) = 9e^{3t} + c_2\left(3te^{3t} + e^{3t}\right) + 3\left(3t^2e^{3t} + 2te^{3t}\right)$$

$$z'(0) = 9 + c_2 = 12 \implies c_2 = 3$$

$$z(t) = 3e^{3t}\left(t^2 + t + 1\right)$$

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