1. [2360/072624 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The set of all vectors of the form $\vec{\mathbf{x}} = \begin{bmatrix} a & b & c & abc \end{bmatrix}^{\mathrm{T}}$, $a, b, c \in \mathbb{R}$, is a subspace of $\mathbb{R}^4$.

   (b) $y'' + 2y' + 3y = \ln t,\ t > 0$, can be solved using the method of undetermined coefficients.

   (c) An undamped harmonic oscillator consisting of a mass of 2 kg and having a circular (angular) frequency of 5 sec^{-1} is subject to an external force of 5 N precisely and only at $t = 5$. The differential equation governing the motion is $2\ddot{x} + 50x = 5\delta(t - 5)$.

   (d) The following system of equations has no equilibrium solutions.

$$\begin{aligned} x' &= x^2 + y^4 + 1 \\ y' &= x^4 + y^2 \end{aligned}$$

   (e) The set $\left\{t^2 + t, 2t, 3t - 1\right\}$ forms a basis for $\mathbb{P}_2$.

   (f) The equilibrium point of the following system is an unstable spiral: $\vec{\mathbf{x}}' = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \vec{\mathbf{x}}$.

   (g) If $\mathbf{Q}$ is a square matrix such that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$, then $\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right)^{-1} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}$.

   (h) The integrating factor for $t^2 y' + 4ty = t^{-2}$ is $\mu(t) = 2t^2$.

SOLUTION:

   (a) **FALSE** Not closed under scalar multiplication. For example, if $\vec{\mathbf{x}} = \begin{bmatrix} 1 & 2 & 3 & 6 \end{bmatrix}^{\mathrm{T}}$, then $2\vec{\mathbf{x}} = \begin{bmatrix} 2 & 4 & 6 & 12 \end{bmatrix}^{\mathrm{T}} \neq \begin{bmatrix} 2 & 4 & 6 & 48 \end{bmatrix}^{\mathrm{T}}$

   (b) **FALSE** It is constant coefficient, but the forcing term, $\ln t$, is not of the proper form.

   (c) **TRUE** $\omega_0 = \sqrt{\dfrac{k}{m}} \implies 5 = \sqrt{\dfrac{k}{2}} \implies 25 = \dfrac{k}{2} \implies k = 50$ and a 5 N external force imparted at a single point in time is modeled as $5\delta(t - 5)$.

   (d) **TRUE** The system has no v nullclines since $x^2 + y^4 = -1$ has no real solutions.

   (e) **TRUE** $W[t^2 + t, 2t, 3t - 1] = \begin{vmatrix} t^2 + t & 2t & 3t - 1 \\ 2t + 1 & 2 & 3 \\ 2 & 0 & 0 \end{vmatrix} = 2(-1)^{3+1} \begin{vmatrix} 2t & 3t - 1 \\ 2 & 3 \end{vmatrix} = 2\left[6t - (6t - 2)\right] = 4 \neq 0 \implies$ we have three linearly independent vectors in a vector space whose dimension is three, implying the vectors form a basis for $\mathbb{P}_2$.

   (f) **FALSE** $\operatorname{Tr}\mathbf{A} = 8, |\mathbf{A}| = 12, (\operatorname{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = 16 > 0 \implies$ unstable, nondegenerate (repelling) node

   (g) **TRUE** $\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right)^{-1} = \mathbf{Q}^{-1}\left(\mathbf{Q}^{\mathrm{T}}\right)^{-1} = \mathbf{Q}^{\mathrm{T}}\left(\mathbf{Q}^{-1}\right)^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\left(\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}$

   (h) **FALSE** The equation has to be in standard form before finding the integrating factor. The proper form is $y' + \dfrac{4}{t}y = t^{-4}$ leading to an integrating factor of t^4.

2. [2360/072624 (44 pts)] Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) (6 pts) Is $\mathbf{A}$ invertible? Justify your answer.

   (b) (6 pts) Is $\mathbf{A}$ in RREF? If not, what is its RREF?

   (c) (6 pts) How many solutions does $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have? Justify your answer.

   (d) (6 pts) Is $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in col $\mathbf{A}$? Justify your answer.

(e) (20 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Write your answer as a single vector.

SOLUTION:

(a) No. $|\mathbf{A}| = 0$.

(b) No. The RREF is $\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$.

(c) Infinitely many. $|\mathbf{A}| = 0$.

(d) No. This is equivalent to asking if $\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix}1\\1\\2\end{bmatrix}$ has a solution. The system is inconsistent since the RREF of the augmented matrix is $\left[\begin{array}{ccc|c}1 & 1 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 1\end{array}\right]$.

(e)

$$\begin{vmatrix}-\lambda & 0 & 0\\1 & 1-\lambda & 0\\0 & 0 & 1-\lambda\end{vmatrix} = (-1)^{3+3}(1-\lambda)\begin{vmatrix}-\lambda & 0\\1 & 1-\lambda\end{vmatrix} = -\lambda\,(1-\lambda)^2 = 0$$

$$\Longrightarrow \lambda = 0,\ \text{algebraic multiplicity } 1;\ \lambda = 1,\ \text{algebraic multiplicity } 2$$

$$\lambda = 0 \Longrightarrow (\mathbf{A} - 0\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}: \left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \Longrightarrow \vec{\mathbf{v}} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\lambda = 1 \Longrightarrow (\mathbf{A} - 1\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}: \left[\begin{array}{ccc|c}-1 & 0 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right] \Longrightarrow \vec{\mathbf{v}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{\mathbf{x}}(t) = c_1\begin{bmatrix}-1\\1\\0\end{bmatrix} + c_2e^t\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3e^t\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \text{apply initial conditions}$$

$$\vec{\mathbf{x}}(0) = c_1\begin{bmatrix}-1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-1 & 0 & 0 & 1\\1 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & -1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 1\end{array}\right]$$

$$\vec{\mathbf{x}}(t) = -1\begin{bmatrix}-1\\1\\0\end{bmatrix} + 2e^t\begin{bmatrix}0\\1\\0\end{bmatrix} + e^t\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2e^t - 1\\e^t\end{bmatrix}$$

■

3. [2360/072624 (40 pts)] A mass/spring system (harmonic oscillator) is governed by the differential equation $\ddot{x} + 8\dot{x} + 25x = f(t)$.

(a) (5 pts) What is $f(t)$ if the oscillator is unforced?

(b) (5 pts) Suppose the system is unforced and the initial displacement is positive. Will the mass pass through its equilibrium position more than once, regardless of the initial velocity? Justify your answer.

(c) (5 pts) Does an $f(t)$ exist such that the total energy of the system remains constant for all time? Justify your answer.

(d) (25 pts) If $f(t) = \begin{cases}0 & 0 \le t < 2\\25 & t \ge 2\end{cases}$ and the mass is resting at its equilibrium position when $t = 0$, use Laplace Transforms to find the equation of motion of the oscillator.

SOLUTION:

(a) $f(t) \equiv 0$

(b) Yes. $b^2 - 4mk = 8^2 - 4(1)(25) = -36 < 0$ so the system is underdamped. Since at least one of the initial conditions is nonzero, there will be oscillatory motion implying that the mass will pass through its equilibrium position more than once.

(c) No. The system is damped so it is nonconservative regardless of what $f(t)$ is.

(d) The IVP is $\ddot{x} + 8\dot{x} + 25x = 25\,\text{step}\,(t-2),\ x(0) = \dot{x}(0) = 0$. Using Laplace transforms we have

$$s^2X(s) - sx(0) - \dot{x}(0) + 8\left[sX(s) - x(0)\right] + 25X(s) = \frac{25e^{-2s}}{s}$$

$$X(s) = \frac{25e^{-2s}}{s\left(s^2+8s+25\right)}$$

$$\frac{25}{s\left(s^2+8s+25\right)} = \frac{A}{s} + \frac{Bs+C}{\left(s^2+8s+25\right)}$$

$$25 = A\left(s^2+8s+25\right) + (Bs+C)s$$

$$s = 0 : 25 = A(25) \implies A = 1$$

$$s = 1 : 25 = (1)(34) + B + C \implies B + C = -9$$

$$s = -1 : 25 = (1)18 + B - C \implies B - C = 7$$

$$B = \frac{\begin{vmatrix} -9 & 1 \\ 7 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{2}{-2} = -1 \qquad C = \frac{\begin{vmatrix} 1 & -9 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{16}{-2} = -8$$

$$x(t) = \mathscr{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s} - \frac{s+8}{s^2+8x+16-16+25}\right)\right\} = \mathscr{L}^{-1}\left\{e^{-2s}\left[\frac{1}{s} - \frac{s+4}{(s+4)^2+9} - \left(\frac{4}{3}\right)\frac{3}{(s+4)^2+9}\right]\right\}$$

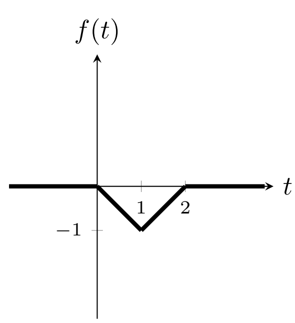
$$= \text{step}(t-2)\mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{s+4}{(s+4)^2+9} - \left(\frac{4}{3}\right)\frac{3}{(s+4)^2+9}\right\}\Big|_{t\to t-2}$$

$$= \text{step}(t-2)\left[1 - e^{-4t}\left(\cos 3t + \frac{4}{3}\sin 3t\right)\right]\Big|_{t\to t-2}$$

$$= \text{step}(t-2)\left\{1 - e^{-4(t-2)}\left[\cos 3(t-2) + \frac{4}{3}\sin 3(t-2)\right]\right\}$$

4. [2360/072624 (20 pts)] The following problems are not related.

(a) (10 pts) The graph of $f(t)$ is shown in the figure. Write $f(t)$ as a single function, not as a piecewise defined function.



(b) (10 pts) Find the Laplace transform of $g(t) = t\,\text{step}(t-1) + \sin(t-\pi)\,\text{step}(t-\pi) - t\delta(t-4)$

SOLUTION:

(a) Begin by writing the function as a piecewise defined function. $f(t) = \begin{cases} 0 & t < 0 \\ -t & 0 \le t < 1 \\ t-2 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$ Then, using step functions we have

$$\begin{aligned} f(t) &= -t\,\text{step}(t) + t\,\text{step}(t-1) + (t-2)\text{step}(t-1) - (t-2)\text{step}(t-2) \\ &= t\,[\text{step}(t-1) - \text{step}(t)] + (t-2)\,[\text{step}(t-1) - \text{step}(t-2)] \end{aligned}$$

(b)

$$\begin{aligned} \mathscr{L}\left\{t\,\text{step}(t-1) + \sin(t-\pi)\,\text{step}(t-\pi) - t\delta(t)\right\} &= \mathscr{L}\left\{t\,\text{step}(t-1)\right\} + \mathscr{L}\left\{\sin(t-\pi)\,\text{step}(t-\pi)\right\} - \mathscr{L}\left\{t\delta(t)\right\} \\ &= e^{-s}\mathscr{L}\left\{t+1\right\} + \frac{e^{-\pi s}}{s^2+1} - (-1)\frac{\mathrm{d}}{\mathrm{d}s}e^{-4s} \\ &= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + \frac{e^{-\pi s}}{s^2+1} - 4e^{-4s} \end{aligned}$$

5. [2360/072624 (22 pts)] Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2}(y-1)^3$.

(a) (2 pts) Is the differential equation autonomous? Why or why not?

(b) (3 pts) Find all equilibrium solutions and determine their stability.

(c) (3 pts) Plot the phase line, correctly depicting all equilibrium solutions.

(d) (2 pts) For any initial value y_0, what is $\lim\limits_{x\to\infty} y(x)$? Justify your answer.

(e) (12 pts) Find the explicit form of the solution of the differential equation that passes through the origin.

SOLUTION:

(a) Yes. There are no explicit instances of the independent variable x, that is, it is in the form $y' = f(y)$.

(b) $y = 1$. If $y < 1, y' > 0$ and if $y > 1, y' < 0$. The equilibrium solution is stable.

(c) Phase line.

$$\begin{array}{c} \downarrow \\ 1\ \bullet \\ \uparrow \end{array}$$

(d) Since the equilibrium solution is stable, $\lim\limits_{t\to\infty} y(t) = 1$ regardless of the initial condition.

(e) Use separation of variables.

$$\begin{aligned} \int -2(y-1)^{-3}\,\mathrm{d}y &= \int \mathrm{d}x \\ (y-1)^{-2} &= x + C \\ (y-1)^2 &= \frac{1}{x+C} \\ y &= 1 \pm \frac{1}{\sqrt{x+C}} \qquad \text{apply the initial condition } y(0) = 0 \\ 0 &= 1 \pm \frac{1}{\sqrt{C}} \\ -1 &= \pm\frac{1}{\sqrt{C}} \qquad \text{must choose minus sign } \implies C = 1 \\ y &= 1 - \frac{1}{\sqrt{x+1}} \end{aligned}$$