APPM 3570/STAT 3100 — Exam 3 — Spring 2024

On the front of your bluebook, write (1) **your name**, (2) **Exam 3**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:__________

1. [EXAM03] (30pts) There are 3 unrelated parts to this question. Justify your answers.

   (a) (10pts) Two candies are selected at random from a jar containing three M&M's©, two Reese's Pieces™ and four Smarties®. Let X and Y be, respectively, the number of M&M's and Reese's Pieces included among the two candies selected. Find $p_{X|Y}(0|1) = P(X = 0 \mid Y = 1)$.

   (b) (10pts) Eight new employees are hired by a company. The company has 10 different divisions and will assign each of the new employees independently and randomly to one of the 10 divisions, what is the expected number of divisions that receive exactly one new employee?

   (c) (10pts) If the random variables X, Y and Z have the means $\mu_X = 4$, $\mu_Y = 9$ and $\mu_Z = 3$, the variances $\sigma_X^2 = 3$, $\sigma_Y^2 = 7$ and $\sigma_Z^2 = 5$ and the covariances $\text{cov}(X, Y) = 1$, $\text{cov}(X, Z) = -3$ and $\text{cov}(Y, Z) = -2$. Find the variance of $W = 2X - 3Y + 4$.

Solution: (a)(10pts) Note that $P(X = 0 \mid Y = 1) = \dfrac{P(X = 0, Y = 1)}{P(Y = 1)}$ where $P(X = 0, Y = 1) = \dfrac{\binom{3}{0}\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} = \dfrac{8}{36} = \dfrac{2}{9}$ and

$$P(Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) = \frac{\binom{3}{0}\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} + \frac{\binom{3}{1}\binom{2}{1}\binom{4}{0}}{\binom{9}{2}} = \frac{8}{36} + \frac{6}{36} = \frac{14}{36} = \frac{7}{18},$$

so

$$P(Y = 0 \mid X = 1) = \frac{P(X = 1, Y = 0)}{P(X = 1)} = \frac{8/36}{14/36} = \frac{8}{14} = \frac{4}{7}.$$

(b)(10pts) Let $Z_i = \begin{cases} 1, & \text{if division } i \text{ gets exactly one new employee,} \\ 0, & \text{else,} \end{cases}$ for $1 \le i \le 10$ and let $Z = \sum_{i=1}^{10} Z_i$ then note that $P(Z_i = 1) = \binom{8}{1}\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^7$ and so

$$E[Z] = E\left[\sum_{i=1}^{10} Z_i\right] = \sum_{i=1}^{10} E\left[Z_i\right] = \sum_{i=1}^{10} P(Z_i = 1) = 10 \cdot \binom{8}{1}\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^8 \approx 0.3444.$$

(c)(10pts) Note that $V(W) = V(2X - 3Y + 4) = V(2X - 3Y)$ and using the fact that $\sigma_X^2 = 3$, $\sigma_Y^2 = 7$ with covariance $\text{cov}(X, Y) = 1$, we have

$$\begin{aligned} V(W) = V(2X - 3Y) &= \text{cov}(2X - 3Y, 2X - 3Y) \\ &= 2^2 V(X) + (-3)^2 V(Y) + 2(2 \cdot -3)\text{cov}(X, Y) = (4 \cdot 3) + (9 \cdot 7) + (-12 \cdot 1) = 63. \end{aligned}$$

2. [Exam03] (40pts) A nut company markets cans of deluxe mixed nuts containing *almonds*, *cashews*, and *peanuts*. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Let X be the weight of almonds in a selected can and Y the weight of cashews. The joint pdf of (X, Y) is given to be

$$f(x,y) = \begin{cases} 8xy, & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \text{ and } y \le x, \\ 0, & \text{else.} \end{cases}$$

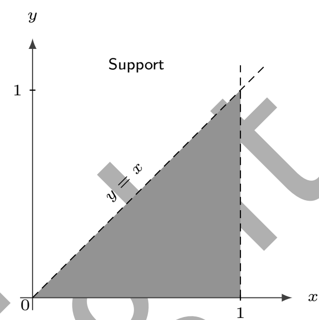
(a) (10pts) Set-up, but *do not solve*, a double integral (or integrals) to find $P(Y < 1 - X)$.

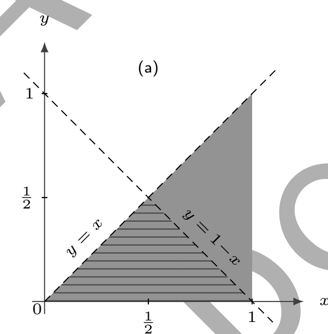
(b) (10pts) Find $f_X(x)$, the marginal pdf for the weight of almonds, X, and find the expectation $E[X]$. (Be sure to define the pdf for all values of $\mathbb{R}$.)

(c) (10pts) Find the conditional probability density $f_{Y|X}(y|x)$ and find the conditional expectation $E[Y|X]$. (Be sure to explicitly specify the domains as appropriate.)

(d) (10pts) Find the expectation $E[Y]$ using the formula $E[Y] = E\big[E[Y|X]\big]$.

Solution:





(a)(10pts) Note that

$$P(Y < 1 - X) = P(Y < X < 1 - Y,\ 0 < Y < \tfrac{1}{2}) = \int_0^{1/2} \int_y^{1-y} 8xy\, dx dy.$$

(b)(10pts) To find the marginal pdf of X, we have, for each $x \in (0,1)$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^x 8xy\, dy = 8x \cdot \left(\frac{y^2}{2} \Big|_0^x \right) = 4x^3,\ 0 < x < 1 \text{ and } 0 \text{ otherwise.}$$

The expecation is

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_0^1 x \cdot 4x^3\, dx = \frac{4x^5}{5}\Big|_0^1 = \frac{4}{5}.$$

(c)(10pts) For each $x \in (0,1)$, we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \dfrac{8xy}{4x^3}, & \text{for } 0 \le y \le x, \\ 0, & \text{else,} \end{cases} = \begin{cases} \dfrac{2y}{x^2}, & \text{for } 0 \le y \le x, \\ 0, & \text{else.} \end{cases}$$

The conditional expectation, for each $x \in (0,1)$, is

$$E[Y|X] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy = \int_0^x y \cdot \frac{2y}{x^2}\, dy = \frac{2}{x^2} \left(\frac{y^3}{3} \Big|_0^x \right) = \frac{2x}{3},\ 0 < x < 1.$$

(d)(10pts) Recall that $E[Y] = E\big[E[Y|X]\big]$ so

$$E[Y] = E\big[E[Y|X]\big] = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x)\, dx = \int_0^1 \frac{2x}{3} \cdot 4x^3\, dx = \frac{8}{3} \cdot \frac{x^5}{5}\Big|_0^1 = \frac{8}{15}.$$

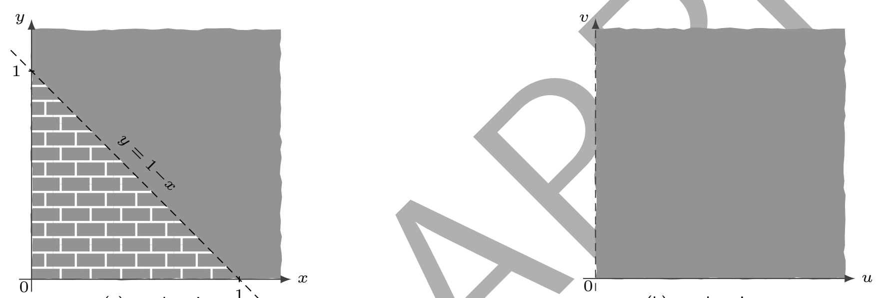
3. [Exam03] (30pts) Suppose X and Y are independent and identically distributed *Exponential random variables* each with parameter $\lambda = 1$.

(a) (10pts) Find the probability $P(X+Y \leq 1)$. Show all work.

(b) (10pts) Compute the joint probability density function of $U = X, V = \dfrac{X}{Y}$. (Be sure to specify the domain.)

(c) (10pts) Find the conditional probability density function $f_{V|U}(v|u)$ and its domain. Are U and V independent? Why or why not?

Solution:



(a) xy-domain

(b) uv-domain

(a)(10pts) First note that, due to independence, and since $X, Y \sim \text{Exp}(\lambda = 1)$, we have

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = (\lambda e^{-\lambda x}) \cdot (\lambda e^{-\lambda y}) = \lambda^2 e^{-\lambda(x+y)} = e^{-(x+y)} \text{ for } x \geq 0 \text{ and } y \geq 0.$$

Note that

$$\begin{aligned}
P(X+Y \leq 1) &= P(0 < X < 1,\, 0 < Y < 1 - X) \\
&= \int_0^1 \int_0^{1-x} e^{-x}e^{-y}\, dydx \\
&= \int_0^1 e^{-x} \left(-e^{-y} \Big|_0^{1-x} \right) dx \\
&= \int_0^1 e^{-x}\left(1 - e^{x-1}\right) dx \\
&= \int_0^1 \left(e^{-x} - e^{-1}\right) dx = (-e^{-1} + 1) - e^{-1} = 1 - \frac{2}{e}.
\end{aligned}$$

(b)(10pts) If $U = X$ and $V = X/Y$, then $X = U$ and $Y = X/V = U/V$ and, furthermore, we have

$$J(x,y) = \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1/y & -x/y^2 \end{vmatrix} = -x/y^2 \Rightarrow |J(x,y)|^{-1} = \frac{y^2}{x} = \frac{(u/v)^2}{u} = \frac{u}{v^2} = |J(u,v)|$$

and note that $x \geq 0 \Rightarrow u \geq 0$ and, due to the denominator restriction, we have $y > 0$ which implies $\frac{u}{v} > 0 \Rightarrow v > 0$. Finally, since $f_{X,Y}(x,y) = e^{-(x+y)}$, we have

$$f_{U,V}(u,v) = f_{X,Y}(u, \tfrac{u}{v}) \cdot \underbrace{|J(x,y)|^{-1}}_{|J(u,v)|} = \begin{cases} e^{-(u+\frac{u}{v})} \cdot \frac{u}{v^2}, & \text{for } u \geq 0,\ v > 0, \\ 0, & \text{else.} \end{cases}$$

(c)(10pts) Since $U = X$, we know that $f_U(u) = e^{-u}$, $u \geq 0$ and 0 otherwise, thus, for each $u \geq 0$, we have

$$f_{V|U}(v|u) = \frac{f(u,v)}{f_U(u)} = \begin{cases} \frac{e^{-u}e^{-u/v}}{e^{-u}} \cdot \frac{u}{v^2}, & \text{for } v > 0, \\ 0, & \text{else,} \end{cases} = \begin{cases} e^{-\frac{u}{v}} \cdot \frac{u}{v^2}, & \text{for } v > 0, \\ 0, & \text{else.} \end{cases}$$

No, the random variables U and V are *not* independent since, for example, $\frac{f(u,v)}{f_U(u)}$ is not strictly a function of v.