APPM/MATH 4/5520

Exam II Review Problems

The exam will be on Thursday, November 10th from 6:30 to 9:00pm in FLMG 155.

Optional Extra Review Session: will be on Wednesday, November 9th from 6 to 8 pm in MUEN E113.

1. Let X_1, X_2 be a random sample from the $N(0, \sigma^2)$ distribution. Find the distribution of

$$\frac{X_1}{|X_2|}?$$

   Name it! (*Hint:* $|X_2| = \sqrt{X_2^2}$.)

2. Let S_1^2 be the sample variance for a random sample of size n_1 from the $N(\mu_1, \sigma^2)$ distribution. Let S_2^2 be the sample variance for an independent random sample of size n_1 from the $N(\mu_1, \sigma^2)$ distribution. Define the pooled variance, S_p^2, as the weighted average

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

   Find the distribution of $(n_1 + n_2 + 2)S_p^2/\sigma^2$.

3. Suppose that a random sample of size 10, taken from the $N(\mu, 3)$ distribution, results in a sample mean of 4.9. Give an 80% confidence interval for the true mean μ.

4. Let $X \sim unif(0, \theta)$. Give a 95% confidence interval for θ.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x; \theta) = \frac{2}{\theta^2} x e^{-(x/\theta)^2} I_{(0,\infty)}(x)$$

   (Note: This is a "Weibull" distribution.)

   (a) Show that $W = 2\sum_{i=1}^{n} X_i^2/\theta^2 \sim \chi^2(2n)$.

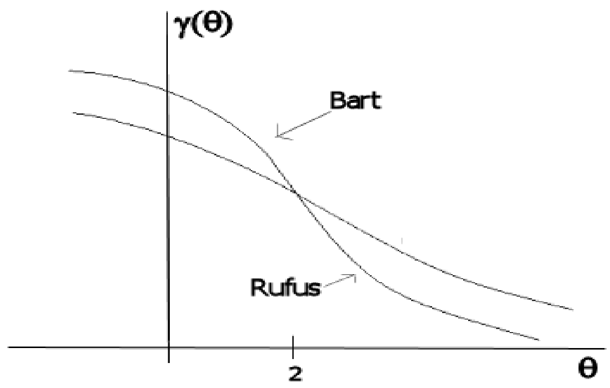
   (b) Use W to derive a $100(1 - \alpha)\%$ confidence interval for θ. (Your answer should involve χ^2 critical values. ie: Use the notation $\chi^2_{\alpha,n}$ to denote the value that cuts off area α to the right on a $\chi^2(n)$ curve.)

6. Consider a random sample $X_1, X_2, \ldots, X_n$ from a distribution with parameter θ. Suppose we want to test the hypotheses

$$\begin{aligned} H_0 &: \theta \geq 2 \\ H_1 &: \theta < 2 \end{aligned}$$

   at level of significance α. Two of your friends, Bart and Rufus, are arguing over who has a better decision rule for performing the test. (That is, they chose to base their tests on different statistics.)

   The two decision rules result in two power functions $\gamma_{Bart}(\theta)$ and $\gamma_{Rufus}(\theta)$ shown in the graph below. (The labels are messed up. They are both pointing to the same curve. I don't have time right now to fix this right now. Please switch Rufus to the other curve!)



(a) Who has the better test?

(b) Where is α on this graph?

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with rate λ. Consider testing the hypotheses

$$\begin{aligned} H_0 &: \lambda = \lambda_0 \\ H_1 &: \lambda < \lambda_0 \end{aligned}$$

(a) Find a test of size α based on the $X_{(1)}$, the minimum value in the sample.

(b) Find the power function for your test from part (a).

8. Let X be a random sample of size 1 from the shifted exponential distribution with rate 1 which has pdf

$$f(x;\theta) = e^{-(x-\theta)}\, I_{(\theta,\infty)}(x).$$

(a) Find a test of size α for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ based on looking at that single value in the sample.

(b) Find the power function for your test.

9. Consider a random sample of size n from the uniform$(0,\theta)$ distribution.

(a) Find the probability that the random interval $(X_{(n)}, 2X_{(n)})$ contains θ.

(b) Find the constant c such that $(X_{(n)}, cX_{(n)})$ is a $100(1-\alpha)\%$ confidence interval for θ.

10. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $N(\mu,\sigma^2)$ distribution. Recall that $\overline{X} \sim N(\mu,\sigma^2)$ and $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$ are independent.

(a) Using the definition of the t-distribution as a ratio of random variables, find the distribution of $\frac{\overline{X}-\mu}{S/\sqrt{n}}$.

(b) Use part (a) to derive a $100(1-\alpha)\%$ confidence interval for μ from scratch.

(c) Derive a $100(1-\alpha)\%$ confidence interval for σ^2.

11. Let $X_1, X_2, \ldots, X_{16}$ be a random sample from the $N(\mu, 9)$ distribution. Find a test of size 0.05 of

$$\begin{aligned} H_0 &: \mu = 3 \\ H_1 &: \mu \neq 3 \end{aligned}$$

based on the sample mean $\overline{X}$.

12. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Weibull distribution with parameters α and β, with γ fixed and known to be 1. Find method of moments estimators for α and β.

13. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x;\theta) = \frac{3}{\theta^3} x^2 \cdot I_{(0,\theta)}(x)$$

where $\theta > 0$. Find the method of moments estimator (MME) of θ.

14. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Pareto distribution with parameter γ. Find the maximum likelihood estimator (MLE) of γ.

15. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution over the interval $(0, \theta)$ for some $\theta > 0$.

    (a) Find the maximum likelihood estimator (MLE) of θ.

    (b) Find an MLE for the median of the distribution. (The median is the number that cuts the area under the pdf exactly in half.)

16. Let $X_1, X_2, \ldots X_n$ be a random sample from the distribution with pdf

$$f(x;\theta,\lambda) = \theta\,\lambda^{\theta}\,x^{-(\theta+1)}\,I_{(\lambda,\infty)}(x)$$

where $\theta > 0$ and $-\infty < \lambda < \infty$.

Find the MLEs for θ and λ.

17. Let $X_1, X_2, \ldots, X_n$ be a random sample from the shifted exp onential distribution with rate 1 which has pdf

$$f(x;\theta) = e^{-(x-\theta)}\,I_{(\theta,\infty)}(x).$$

    (a) Find the MME of θ.

    (b) Find the MLE of θ.

    (c) To simplify further computations, show that $\overline{X} - \theta \sim GAMMA(n, n)$ and that $X_{(1)} - \theta \sim exp(rate = n)$. (Hint: Personally, I think it is easiest to use mgf's for the first problem and cdf's for the second.)

    (d) Are your two estimators unbiased? (Hint: Use part (c). For example, if you are trying to find $\mathsf{E}[X_{(1)}]$, you could write this as $\mathsf{E}[X_{(1)} - \eta + \eta] = \mathsf{E}[X_{(1)} - \eta] + \eta$.)

    (e) Find the MSE for each of your two estimators.

    (f) At this point, you should have found that your MME was unbiased and that your MLE was biased. Can you add or subtract something from your biased MLE to make an unbiased estimator, $\hat{\theta}_3$, for θ? Find the MSE for this new estimator and decide which is the best (with respect to MSE) estimator of all three.

18. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on the interval $(0, \theta)$.

Consider testing

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta < \theta_0 \end{aligned}$$

    (a) Find a test of size α based on the $X_{(n)}$, the maximum of the sample.

(b) Find a test of size α based on the $X_{(1)}$, the minimum of the sample.

19. Let $X_1, X_2, \ldots X_n$ be a random sample from the binomial distribution with parameters m and p with m fixed and known. Find the MLE of $P(X = 0)$.

20. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)} I_{(0,\infty)}(x), \qquad \theta > 0.$$

   (a) Find the MLE $\hat{\theta}_n$ of θ.

   (b) What is $\lim_{n\to\infty} \mathsf{E}[\hat{\theta}_n]$?

21. Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric distribution

$$f(x;p) = p(1-p)^x I_{\{0,1,\ldots\}}(x), \qquad 0 < p < 1.$$

   (a) Find the MME for p.

   (b) Find the MLE for p.

   (c) Find the MLE for $\ln p$.

   (d) Find the asymptotic distribution of $\hat{p}$ (MLE).

22. Consider a random sample of size n from the distribution with pdf

$$f(x;\theta) = \frac{(\ln\theta)^x}{\theta x!} \cdot I_{\{0,1,\ldots\}}$$

   with $\theta > 1$.

   (a) Find the CRLB for θ.

   (b) Find the CRLB for $(\ln\theta)^2$.

   (c) Find the CRLB for $P(X_1 \le 1)$.

23. (a) Show that $\mathsf{E}\left[\frac{\partial}{\partial\theta} \ln f(\vec{X};\theta)\right] = 0$.

   (b) What is $Var\left[\frac{\partial}{\partial\theta} \ln f(\vec{X};\theta)\right]$ in terms of the Fisher Information?

24. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential rate θ distribution.

   (a) Find the CRLB for θ. What is it bounding?

   (b) Find the CRLB for $1/\theta$.

   (c) Find the MLE of $P(X > 1)$.

   (d) Find the CRLB for $P(X > 1)$.

25. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)} I_{(0,\infty)}(x), \qquad \theta > 0.$$

   Find the asymptotic distribution of $\hat{\theta}_n$, where $\hat{\theta}_n$ is the MLE for θ.

26. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf $f(x;\theta)$. Consider a statistic $S = s(\vec{X})$.

(a) What does it mean for S to be sufficient for θ?

(b) What does it mean for S to be complete for θ?

27. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $geom_0(p)$ distribution. Show, using the definition of a sufficient statistic, that $S = \sum_{i=1}^{n} X_i$ is sufficient for p.

28. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on the interval $[\theta_1, \theta_2]$. Find a (vector of) sufficient statistic(s) for estimating θ_1 and θ_2. If your answer was not two-dimensional, find a two-dimensional sufficient statistic for estimating θ_1 and θ_2.

29. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution with parameter λ.

    (a) Find a one-dimensional sufficient statistic for this model.

    (b) Use the Rao-Blackwell Theorem to find an unbiased estimator of $\tau(\lambda) = e^{-\lambda}$ based on your sufficient statistics from part (a).

30. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x; \theta) = e^{\theta - x} I_{(\theta,\infty)}(x).$$

    Find a complete and sufficient statistic for estimating θ.

31. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $Pareto(\gamma)$ distribution.

    Find a complete and sufficient statistic for estimating γ.