1 Introduction

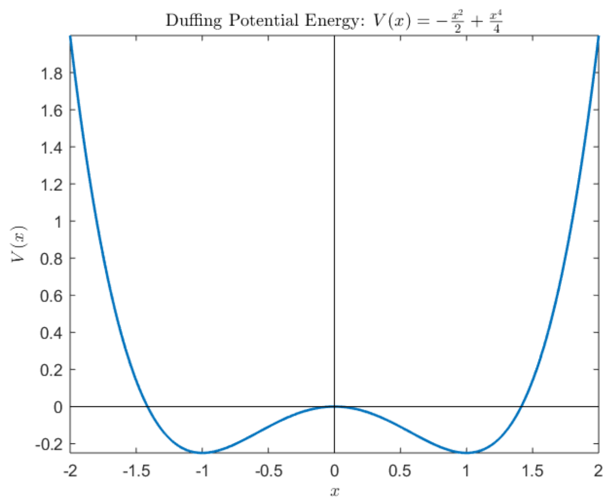
The Duffing Equation is an externally forced and damped oscillator equation that exhibits a range of interesting dynamic behavior in its solutions. While, for many parameter values, the solutions of the system represent a mass-spring system whose response to displacement from equilibrium is characterized by a restoring force exhibiting both linear and cubic features, the system's solutions readily transition to chaotic behavior. This report will explore the Duffing Equation in both the chaotic and non-chaotic regimes.

2 The Non-Chaotic Duffing Equation

For certain parameter values, the Duffing Equation reasonably describes a mass trapped in a double well potential, which is equivalent to saying that the system's response to displacement from equilibrium comes from a quartic potential function. This energy function (or system response) is of the form

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$$

and can be visualized as



(From this plot of the potential energy, the double-well feature of the potential is utterly clear.) Furthermore, it is clear that the extrema of the potential appear as minimum and occur at $x = \pm 1$. The values of the minima are $V(x = \pm 1) = -\frac{1}{4}$. We will see later that the positions of the minima of this potential correspond to stable equilibrium solutions in the Duffing Equation that is derived from this potential.

Using the potential energy function $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ and Newton's Laws, a special case of the Duffing Equation can be derived and has the form

$$\ddot{x} - x + x^3 = 0$$

This second order, nonlinear differential equation is both undamped and unforced. Under the variable transformation $v = \dot{x} \implies \dot{v} = \ddot{x}$ an equivalent system of equations can be derived (see Appendix 1):

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = x - x^3$$

We can examine the solutions of this system of equations in a straightforward manner by finding the nullclines and equilibrium solutions of the system and then plotting these along with the system's phase portrait. Recall that nullclines of a system of differential equations are obtained by restricting one direction of motion in the system:

$$\text{Let } \dot{x} = 0 \implies v(t) = 0$$

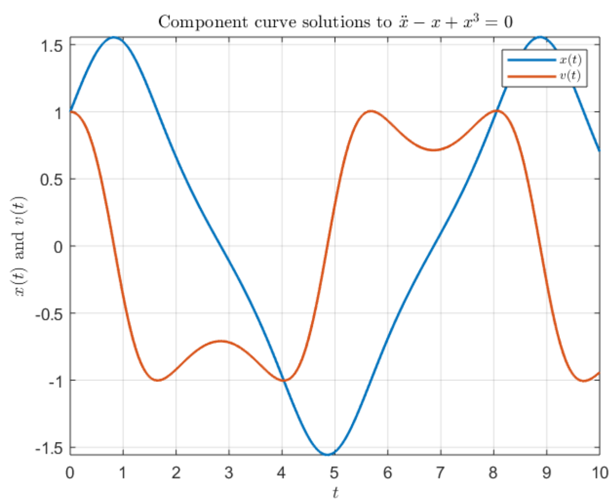
$$\text{Let } \dot{v} = 0 \implies x - x^3 = 0 \implies x = 0 \text{ or } x = \pm 1$$

The vertical nullcline is $v(t) = 0$, while the horizontal nullclines are $x(t) = 0$, $x(t) = \pm 1$. The intersections of vertical and horizontal nullclines give rise to the equilibrium solutions of the system, wherein both directions of motion are required to be 0. The system exhibits three equilibrium solutions of the form

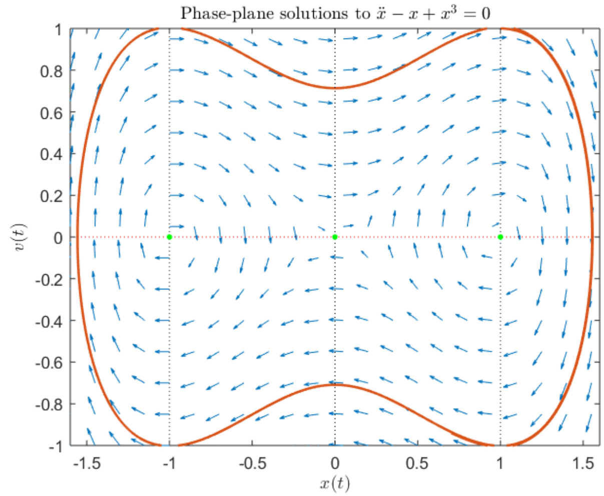
$$(x_{eq}, v_{eq}) = (0, 0), \quad (x_{eq}, v_{eq}) = (1, 0), \quad \text{and} \quad (x_{eq}, v_{eq}) = (-1, 0)$$

Examining the directions of motion along each nullcline and across each equilibrium solution, we expect the system's solutions to orbit the equilibrium solutions at $(\pm 1, 0)$ and diverge away from the equilibrium solution at $(0, 0)$.

To visualize the behavior of these solutions more clearly, let the system start from the initial state $x(0) = v(0) = 1$ (so that both the position and potential energy of an arbitrary particle in the system are 1) and numerically solve the system of equations over $t \in [0, 10]$ with the Matlab solver `ode45`. Examining first the two components of the solution $x(t)$ and $v(t)$, we see that the solutions evolve periodically (or pseudo-periodically) with time.



To examine the influence of the solution components on one another, the phase-plane solution of the system was overlaid with the phase-portrait and the nullclines and equilibria of the system.



From the phase-plane solution, we see that the parametric curve does in fact orbit the set of equilibrium solutions. The equilibrium $(0, 0)$ seems to be unstable while the other equilibria $(\pm 1, 0)$ appear to be centers.

3 Transition to Chaos

We saw in the special case of the Duffing Equation (above) that the phase-space solutions conserved energy and resulted in solution curves that followed a single, exact trajectory without deviation. To eliminate conservation of energy (and to allow the potential for chaotic behavior in the Duffing system) two additional terms must be included in the system. A term, $\delta\dot{x}$, must be included to allow *damping* in the system and a term, $\gamma\cos(\omega t)$, must be included to allow *external forcing* of the system. The result of including these terms is the general Duffing Equation:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$

This equation can be characterized as a second order, nonlinear oscillator with constant coefficients. Each term and/or parameter in this more general equation an be understood in the following way:

- There is periodic external forcing that comes from the term $\gamma\cos(\omega t)$. The parameter γ is the strength of the driving force and ω is the frequency of forcing.
- The term αx is a classical restoring force that follows Hooke's Law (where α is a linear "stiffness" term that is equivalent to a classical spring constant). Meanwhile, the term βx^3 represents a cubic restoring force that controls the nonlinear response of the system. This often leads to an increase in the "stiffness" of the spring since it deviates from classical harmonic motion.
- The term $\delta\dot{x}$, $\delta \geq 0$, represents linear damping in the mass-spring system. (The term $\dot{x}$ is the velocity of the system.)
- The term $\ddot{x}$ is the acceleration of the system under the assumption that the system has mass $m = 1$.

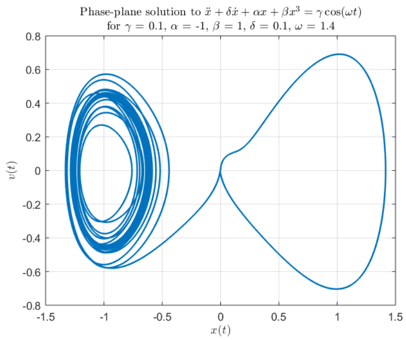
As in the section above, it can be very straightforward to analyze the solutions to such an equation if we first convert the nonlinear equation to a system of first order differential equations. Under the variable transformation $v = \dot{x} \implies \dot{v} = \ddot{x}$, this results in the system (see Appendix 1)

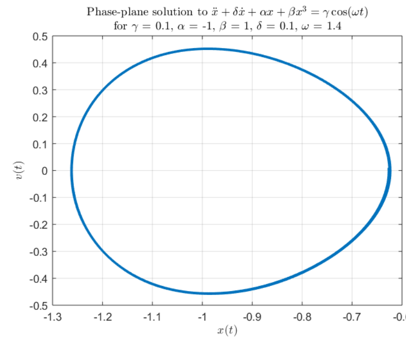
$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\delta v - \alpha x - \beta x^3 + \gamma\cos(\omega t)$$

As discussed above, γ represents the strength of the external driving force of the nonlinear system. Increases the driving force will push the system from deterministic dynamics to chaotic dynamics that cannot be predicted exactly. Investigating the behavior of the system as γ increases yields interesting results that are made most clear if we study both the beginning and ending behavior of each solution obtained for various values of γ.

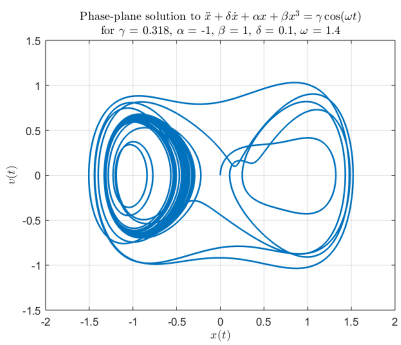
To begin, let $\gamma = 0.1$ and consider the range of time values $t \in [0, 200]$. Plotting both the full parametric

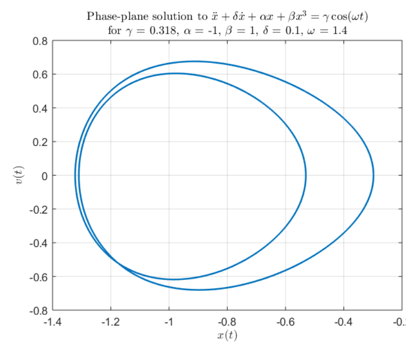




solution and the tail of the solution, we see that the full solution appears to exhibit very strange and unpredictable behavior. However, once the end behavior of the solution is examined more closely we see that the solution approaches a single orbit with period $T = 2\pi/\omega$.

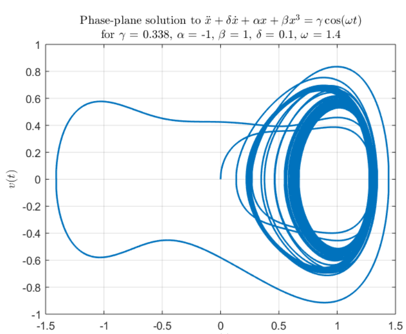
Let $\gamma = 0.318$ and consider the range of time values $t \in [0, 800]$. Observe that increasing both the

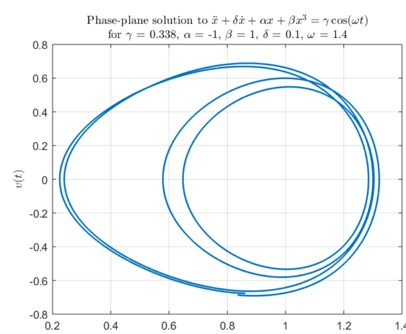




magnitude of the driving force and the interval t over which the solution is computed leads to phenomena similar to those in the previous solution plots. The initial behavior of the solution is unpredictable but restricted to a set region in phase space. However, we see that the end behavior of the solution approaches a simple curve that is composed of two nested orbits. These orbits demonstrate "period doubling" in the solution wherein the period of the solution is $T = 4\pi/\omega$.

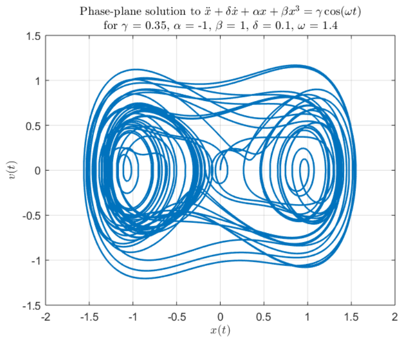
Again increase $\gamma = 0.338$ and consider the range of time values $t \in [0, 2000]$. Further increasing the

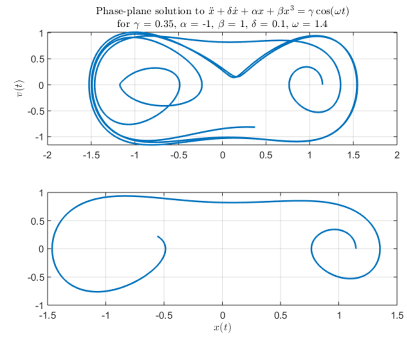




magnitude of the driving force and the length of the t interval reveals the same pattern of behavior as in the previous two cases. In this case, the end behavior of the solution gives rise yet again to period doubling (as can be seen in the four nested orbits of the solution) and the period is $T = 8\pi/\omega$.

Finally, let $\gamma = 0.35$ and consider the range of time values $t \in [0, 3000]$. This final increase in the driving





force γ reveals a very different type of behavior in the Duffing system. While the initial behavior of the other solutions appeared just as unpredictable as this final solution, the final behavior of the other solutions settled down to a single set of nested orbits that reveal bifurcation in the system. However, the final parameter values reveal that the system has transitioned from the phase space where bifurcation occurs into a region of chaos. This does not allow for the system to eventually reach a stable, fixed behavior but instead the solutions continue to move through the phase-space in an unpredictable fashion.

4 Conclusion

This report investigated the Duffing Equation for a range of parameter values. We found, due to energy conservation, that the Duffing Equation is unable to exhibit chaos when the oscillator is undamped and unforced (that is, $\delta = \gamma = 0$). To allow for chaos, energy conservation is eliminated by including both a damping and an external forcing term. Then we see, as the magnitude of external forcing is increased, the system moves through a region of period-doubling bifurcations and then transitions to a chaotic regime. The transition to chaos appears to occur between $\gamma = 0.338$ and $\gamma = 0.35$.

5 Appendix 1

- Conversion of the Special Case Duffing Equation to a System of ODEs:

$$\text{Let } \dot{x} = v \implies \ddot{x} = \dot{v}$$
$$\ddot{x} - x + x^3 = 0$$
$$\implies \dot{v} - x + x^3 = 0$$
$$\implies \dot{v} = x + x^3$$

The system of equations is then

$$\begin{cases} \dot{x} = v \\ \dot{v} = x - x^3 \end{cases}$$

- Conversion of the Duffing Equation to a System of ODEs:

$$\text{Let } \dot{x} = v \implies \ddot{x} = \dot{v}$$
$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$
$$\implies \dot{v} + \delta v + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$
$$\implies \dot{v} = -\delta v - \alpha x - \beta x^3 + \gamma\cos(\omega t)$$

The system of equations is then

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\delta v - \alpha x - \beta x^3 + \gamma\cos(\omega t) \end{cases}$$

6 Appendix 2

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...  Matlab code would go here!  ...
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