Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 2009

Notice. Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work!	1. ___ 2. ___ 3. ___ 4. ___ 5. ___ **TOTAL.** ___

STUDENT NUMBER: ________________

1. Let $X_1, \ldots, X_{100}$ be i.i.d. Exponential random variables with parameter $\lambda = 1$ i.e. $\mathbb{P}[X_i \leq t] = 1 - \exp(-t)$, for all $t \geq 0$.

   (a) For a given $k \geq 1$, what is approximately the probability of the event

   $$\sum_{i=1}^{100} X_i \leq 100 \cdot \left(1 + \frac{k}{10}\right)?$$

   Leave your answer in terms of Φ, the cumulative distribution function of a standard Normal distribution.

   (b) Use the Berry-Esseen inequality (see theorem below) to determine the smallest value of k for which the error in your approximation above can be guarantee to be less or equal to 1%. Simplify your answer!
   Theorem. *(Berry-Esseen inequality.) If $Y_1, \ldots, Y_n$ are i.i.d. random variables with finite mean μ and finite variance $\sigma^2 > 0$ then, for all $t \in \mathbb{R}$, one has that:*

   $$\left| \mathbb{P}\left[\sqrt{n} \cdot \frac{\bar{Y} - \mu}{\sigma} \leq t \right] - \Phi(t) \right| \leq \frac{32}{(1+|t|)^3 \sqrt{n}} \cdot \mathbb{E}\left| \frac{Y_1 - \mu}{\sigma} \right|^3.$$

2. The first part in this problem may be useful for the second part.

   (a) Let $U_1, U_2, \ldots, U_n, V_1, \ldots, V_{n-1}$, with $n \geq 2$, be a random sample of uniform random variables in the interval $[0, 1]$. In what follows $U_{(i)}$ and $V_{(j)}$ respectively denote the order statistics associated with the U- and V-sample. For instance:

   $$\begin{aligned} U_{(n)} &= \max\{U_1, \ldots, U_n\}; \\ V_{(1)} &= \min\{V_1, \ldots, V_{n-1}\}. \end{aligned}$$

   For a given $0 < u < 1$, show that the conditional distribution of $(U_{(1)}, \ldots, U_{(n-1)})$ given that $U_{(n)} = u$ is the same as the distribution of $(u \cdot V_{(1)}, \ldots, u \cdot V_{(n-1)})$.

   (b) Now suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution that has a continuous cumulative distribution function F. Find the probability density function of

   $$\frac{1 - F(X_{(2)})}{1 - F(X_{(1)})}.$$

(THREE MORE PROBLEM ON THE BACK!)

3. Let $N \geq 0$ be an unknown integer and let $X_1, \ldots, X_n$ be a random sample from the distribution with

$$P(X_i = k) = \begin{cases} \frac{1}{2N+1} & , \quad k = 0, \pm 1, \pm 2, \ldots, \pm N; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

(a) Show that $M = \max\{|X_1|, \ldots, |X_n|\}$ is a sufficient statistic for N.

(b) Show that M is a complete statistic for N.

(c) Determine the uniformly minimum variance unbiased estimator (UMVUE) for N. Simplify your answer!
HINT FOR PART (C). Determine constants a, b, c such that $a \cdot |X_1| + b \cdot [\![X_1 = 0]\!] + c$ is unbiased for N, where $[\![X_1 = 0]\!]$ is notation for the indicator function of the event $[X_1 = 0]$.

4. Suppose that X_1 is a random sample of size 1 from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{2\theta} & , \quad \text{if } 0 < x \leq \theta; \\ \frac{\theta}{2x^2} & , \quad \text{if } x \geq \theta; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

For which values of $0 < \alpha < 1$ does $H_0 : \theta = 1$ versus $H_1 : \theta \geq 3$ admit a unique uniformly most powerful (UMP) test of size α? Specify the rejection region associated with each of those tests.

5. Let $0 < \mu < \lambda$ be real constants and consider an M/M/1 queueing system with arrival rate μ and service rate λ, which is initially empty. Suppose that the server is turned ON and OFF according to the following rules:

- it remains OFF as long as the number of customers in the system is less than 2,
- it is turned ON as soon as the number of customers in the system becomes 2 and then remains ON until completely emptying the system of customers.

Based on the above, please respond:

(a) Model the number of customer in the system as a continuous time Markov chain with state space $\{0, 1', 1, 2, 3, \ldots\}$, where $1'$ represents the configuration of having exactly one customer in the system while the server is OFF, and state 1 represents the same but while the server is ON. Represent the rate transition matrix of the chain as a directed graph with weighted edges.

(b) Show that this system has a stationary distribution and determine it explicitly.

(c) After a long time of operation, what is the probability that a new customer encounters the queue empty?