Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 2011

Notice: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____
Total ____

Student Number ____________________

1. Flip k fair coins simultaneously. Afterwards, remove all the coins which came up "Heads". Repeat the process, by flipping the remaining coins simultaneously and removing all coins that come up heads, until there are no more coins. What is the expected number of simultaneous flips?

2. Suppose that X has the Poisson distribution with parameter λ and that $Y|X = x \sim binomial(x + 1, p)$.

   (a) Find the covariance between X and Y. Call it ω.

   (b) Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be n independent copies of (X, Y). Find the maximum likelihood estimator of ω. Call it $\widehat{\omega}$.

   (c) What is the asymptotic distribution of $\widehat{\omega}$? (Name the distribution and define the parameters but do not actually compute the parameters.)

3. Suppose that $X_1, X_2, \ldots, X_n$ are indepent and identically distributed Bernoulli random variables with parameter p. Assume that $n \geq 2$.

   (a) Find a complete and sufficient statistic for p.

   (b) Find an unbiased estimator of $\tau(p) = p^2$.

   (c) Find the UMVUE (uniformly minimum variance unbiased estimator) of $\tau(p)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $uniform(\theta, \theta + 1)$ distribution with $\theta \geq 0$.

   (a) Find the joint probability density function of $X_{(1)}$ and $X_{(n)}$, the minimum and maximum, respectively, of the sample.

   (b) Consider testing the hypotheses

   $$H_0 : \theta = 0 \qquad \text{versus} \qquad H_1 : \theta > 0$$

   using the rule

$$\text{“reject } H_0 \text{ if } X_{(1)} \geq c \text{ or } X_{(n)} \geq 1\text{”}$$

Determine c so that the test will have size α.

(c) Let $\alpha = 0.10$. Find the necessary sample size so that the test will have power at least 0.80 if $\theta \geq 0.5$.

5. Consider a continuous time Markov process $\{X(t)\}_{t \geq 0}$ on the state space $\{0, 1, 2, \ldots\}$ with stationary probabilities $\{\pi_0, \pi_1, \pi_2, \ldots\}$. Suppose that, when currently in state i, the process will jump to state j after an exponential amount of time with rate q_{ij} and that all exponential times are independent.

   Assume that $X(0) = 0$.

   (a) Let ν be the rate of departure from state 0. Write ν in terms of the q_{ij}.

   (b) Let Y be the time of the first exit from state zero. Find the distribution of Y.

   (c) Starting from 0, let R be the time of the first return to state zero. What is $\mathsf{E}[R]$?

   (d) Let T be the first time that the process has been in state 0 for at least τ units (continuous) of time. Show that

$$\mathsf{E}[T] = \frac{1}{\pi_0 \nu} \left(e^{\nu\tau} - 1 \right).$$

   (Hint: Condition on the time of the first exit from state 0.)