Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2010

Notice: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____

Total ____

Student Number ____________________________

1. There are $k+1$ coins in a box labeled 0 through k. Coin i, when flipped, will result in "heads" with probability i/k, for $i = 0, 1, \ldots, k$. A coin is random selected from the box and repeatedly flipped. ESTIMATE all of the following probabilities for large k.

   (*Hint: You may find it useful at some point in your solution to think about the Beta probability distribution!*)

   (a) If the first n flips are all heads, find the conditional probability that the $(n+1)$st flip is also heads?

   (b) If the first n flips resulted in r heads and $n-r$ tails, show that the probability that the $(n+1)$st flip results in heads is $(r+1)/(n+2)$.

2. Let U be a uniform random variable on the interval $(0,1)$. Let c be a constant such that $0 < c < 1$. Let V be a continuous random variable with some distribution on $(0,1)$ that is independent of U.

   (a) Show that $\min\left(\frac{U}{c}, \frac{1-U}{1-c}\right)$ has a $uniform(0,1)$ distribution.

   (b) Find
   $$P\left(\min\left(\frac{U}{V}, \frac{1-U}{1-V}\right) < c\right).$$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean μ and variance 1. Suppose that we want to estimate $\tau(\mu) = P(X_1 > 0)$. For the following, you may leave your answers in terms of $\Phi(\cdot)$, the cdf for the standard normal distribution.

   (a) Show that $X_1 - X_2$ and $X_1 - \overline{X}$ are independent of $\overline{X}$.

   (b) Give an unbiased estimator of $\tau(\mu)$.

(c) Find the UMVUE (uniformly minimum variance unbiased estimator) of $\tau(\mu)$.

(d) Find the MLE (maximum likelihood estimator) of $\tau(\mu)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on the interval $(\theta, \theta + 1)$. Suppose that we wish to test the hypothesis $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ using the test

$$\text{reject } H_0 \text{ if } \; X_{(1)} > 1 \; \text{ or } \; X_{(n)} > c$$

where c is a constant to be determined.

(a) Find c such that the test will have level of significance α.

(b) Find the power function for your test.

(c) Is your test the UMP (uniformly most powerful) test? Explain.

5. Starting at time 0, satellites are launched at times of a Poisson process with rate λ. Suppose that each satellite, once launched, has a lifetime, independent of all others and of the launch process, that has cdf F and mean μ. Let $X(t)$ be the number of launched and working satellites at time t.

(a) Find the distribution of $X(t)$.

(b) Let $t \to \infty$ to show that the limiting distribution is Poisson($\lambda\mu$).

Hint 1: *Given that n satellites have launched, consider writing the surviving satellites at time t as a sum of indicators.*

Hint 2: *Note that, for a random sample $X_1, X_2, \ldots, X_n$ with order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, $\sum_{i=1}^n X_i = \sum_{i=1}^n X_{(i)}$.*

Hint 3: *Recall that, for a nonnegative random variable X, $\mathsf{E}[X] = \int_0^\infty P(X > x)\,dx$.*