Solving Systems of Differential Equations

1 Solving Systems of Differential Equations

We know how to use `ode45` to solve a first order differential equation, but it can handle much more than this. We will now go over how to solve systems of differential equations using Matlab. Consider the system of differential equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{1}{5}y_2 - \sin(y_1) \end{aligned}$$

We would like to solve this forward in time. To do this, we must first create a function M-File that holds the differential equation. It works exactly how the function M-file works for solving a first-order differential equation, except we must treat our variables (except time) as vectors instead of scalars as we did before. The function M-File for this differential equation should be saved as `system_ex.m` and looks like

```
function yprime = system_ex(t,y)
yprime = zeros(2,1);
yprime(1) = y(2);
yprime(2) = -1/5*y(2)-sin(y(1));
```

See how `y` is a vector, where `y(1)` is associated with y_1 and `y(2)` is associated with y_2? The same is true of `yprime`, where `yprime(1)` is associated with y_1 and `yprime(2)` is associated with y_2. That's all there is to it!

Now we'd like to solve the differential equation with initial conditions $y_1(0) = 0$ and $y_2(0) = 3$ forward in time, lets say $t \in [0, 40]$. The command is just the same as we have used before, except we need to give it a vector of initial conditions instead of just a scalar. In the command window, type

```
[t,y] = ode45(@system_ex,[0,40],[0,3])
```

The system has been numerically solved. Looking in the workspace, you see we now have two variables. t holds all the time steps while y is a matrix with 2 columns. The first column of the matrix is all the y_1 values and the second column is all the y_2 values. You can plot these against time to see the solution of each variable, or plot them against each other to generate solutions in the phase plane:

```
plot(t,y(:,1))
plot(t,y(:,2))
plot(y(:,1),y(:,2))
```

Try this with some more initial conditions.

2 Global Variables

Sometimes, we would like to have a parameter inside our function m-file. To do this, we declare a global variable, since it's hard to pass these using `ode45`. Say we now have the system:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= ay^2 - \sin(y_1) \end{aligned}$$

where a is a parameter. In the command window, type:

```
global a
```

and in the `system_ex.m` file, change it to

```
function [yprime] = system_ex(t, y)
global a
yprime(1,1) = y(2);
yprime(2,1) = a*y(2)-sin(y(1));
```

In the command window, set a equal to whatever value you'd like, and plot the solutions using `ode45`. You can see that the value is automatically changed in `system_ex.m` whenever you change it in the command window.

Alternatively, instead of using global variables we could change `system_ex.m` to:

```
function [yprime] = system_ex(t,y,a)
yprime(1,1)=y(2);
yprime(2,1)=a*y(2)-sin(y(1));
```

and in the command window type:

```
[t,y] = ode45(@system_ex,[0,40],[0,3],[],-1/5)
```

3 Contour Plots

Matlab can generate contour plots quite easily. First we create a mesh using `meshgrid`. Then we use the contour command to plot the contours of the given equation. If we wanted to plot the contours for the equation of a circle $x^2 + y^2$ for values of x and y in the unit circle, we type

```
[x,y]=meshgrid(-1:0.01:1,-1:0.01:1);
contour(x,y,x.^2+y.^2,20)
```

Type help contour to see all the optional parameters.

4 Homework #10

Solve the system of equations

$$\begin{aligned} x'(t) &= -\sin(x(t)) + y(t) \\ y'(t) &= -\cos(x(t)) - \frac{1}{5}y(t) \end{aligned}$$

using `ode45`, over the time interval $t \in (0.40)$ with initial conditions $x(0) = 4$ and $y(0) = 0$. Then, plot y against x. You should observe the trajectory approaching an equilibrium , i.e. a single point (x_0, y_0), in this space. (Hint: Generally, we plot y against t using `plot(t,y)`. How would we plot y against x?)