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Continuous Rendezvous Games and Their Departure and Wait times

Joongsan Hwang
University of Colorado Boulder

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Department of Economics

University of Colorado Boulder
Boulder, Colorado 80309

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ESTRAGON	Let's go.
VLADIMIR	We can't.
ESTRAGON	Why not?
VLADIMIR	We're waiting for Godot.

— Samuel Beckett, *Waiting for Godot*

Rendezvous where two people meet each other are common in reality. This paper solves for pure strategy Nash equilibria of these games for continuous distributions of the players' travel times. For the case where players can depart whenever they want, I find Nash equilibria where meeting probability is 1. For the case where players may find themselves unable to depart as early as they want, I find Nash equilibria where players' departure times and decisions vary depending on their start times. In these Nash equilibria, start time variation causes low meeting chance. if players compensate each other for waiting for the other player, they might increase meeting chance and expected utility. Players may also increase meeting chance by agreeing to not depart early but wait moderately.

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1 Introduction

Suppose Alice is meeting Bob at a restaurant at 6 PM. On average, it takes Alice about an hour to get to the restaurant from home. Bob comes from the opposite direction to the restaurant and also takes about an hour to arrive there. When should Alice depart for the restaurant from home? When does Alice think Bob will depart? This is before the advent of cell phones. After Alice arrives at the restaurant, what should she do if she cannot find Bob? Should she wait for him to arrive? How long should she wait? These are some of the questions and decisions that people attempting to rendezvous need to consider and make. What makes this game complex is that there are many stochastic factors such as road conditions involved. People cannot just start for the meeting place and arrive exactly when they want to. This means that a trivial solution like both people agreeing to meet exactly at 6 PM and always doing so cannot happen in real life. In reality, people factor in unexpected occurrences into their decisions. For example, for an important meeting, people depart early for the meeting place not to be late or wait longer even if the other person does not arrive on time.

This paper is the first to provide Nash equilibrium solutions to such two-player rendezvous games with temporal uncertainties. Through my model, I analyze the strategic interactions in the rendezvous game. Despite the fact that such games of rendezvous are played out very frequently in real life, there is no rigorous equilibrium solution to this game that is founded on game theory.

In my model, players have conflicting incentives, an incentive for cooperation and an incentive for exploitation. First, players are incentivized to cooperate by coordinating their departure times to synchronize their arrival times. This way, the players should meet without waiting long for each other. However, players are also incentivized to exploit each other by departing later and increasing the other player's wait time. A player employing such a strategy is trying to make the other player come first and wait for her. If the strategy is successful, the player can meet the other player while shortening her wait time at the expense of lengthening the other player's wait time.

An attractive feature of my model is that it also provides insights to games in other settings. There are a variety of settings where players attempt to coordinate but are sometimes forced to wait for each other in face of uncertainties. I will discuss two examples. The first case is a supply-chain with an upstream firm and a downstream firm. In this case, the downstream firm needs the upstream product to make the downstream firm's sale. The downstream firm might be a retailer that needs to stock the upstream firm's product. It might also be a manufacturer that needs parts, intermediate goods or some other input from the upstream firm to make its product. The firms sign a contract under which the upstream firm delivers its product to the downstream firm till a specific time.

If the downstream firm receives the upstream products too late, its own sales will also be delayed. The downstream firm may have to make some preparations before the firm can use the upstream product. For instance, it might have to make stocking space for the upstream product or acquire other inputs that are used in conjunction with the upstream product. The downstream firm needs to decide when to perform such preparations. On the other hand, the upstream firm may run into unexpected difficulties during production which may hinder it from delivering its products to the downstream firm on time. The upstream firm also needs to consider that it will be penalized for delivering its products too late.

We can also think of service reservations between a business and a client. Establishments such as restaurants often take in reservations from clients and agree to service them at a particular time. Firms are unsure exactly when the client will arrive. While waiting for the client, firms might not be able to service other clients. Restaurants need to keep the reserved table empty. Clients know that if they arrive too early, the firm might not be able to service them immediately because the firm might be busy with other clients or unprepared to handle the new client. They also know that if they arrive too late, they might forfeit the reservation.

Section 4 solves for the Nash equilibria of the model under two different assumptions. Subsection 4.1 assumes that there is no start time variation. Here, I find the necessary and sufficient conditions for pure strategy Nash equilibria. In all these pure strategy Nash equilibria, players wait till the other player comes with probability 1 and thus, the meeting probability is 1. Subsection 4.2 assumes that start times are uniformly distributed for both players. In the subsection's case, I find pure strategy Nash equilibria characterized by two parameters, $\underline{s}$ and $\bar{s}$. These parameters respectively represent the earliest time at which players might depart and the latest time at which players might depart, according to the players' strategies. So the parameters describe how start time variation makes the players' departure times vary.

In subsection 4.1, for given departure times of the players, once the lower bounds on the player's value of the meeting are satisfied, players' values of the meetings can be arbitrarily higher in the pure strategy Nash equilibria. It is not necessary the player with the comparatively higher value of meeting that departs first in the pure strategy Nash equilibria. In the context of meetings involving the head of states, this means that the heads can deliberately depart late for a meeting and have the others wait for them.

In subsection 4.2, the Nash equilibria have low meeting probability because players do not always come nor wait for each other. If players compensate each other for arriving early and waiting, players might increase meeting probability and both their expected utilities. When monetary compensations are difficult to implement, non-monetary compensations such as agreeing that "the person who arrives late pays for the meal" can work in their place.

However, unilateral punishments for late arrival that go beyond compensation may decrease social welfare by harming the player who arrives earlier than the other player to avoid punishment. Given this, I argue that in context of supply chains, such punishments can be avoided by only allowing liquidated damages provisions. Another way that players can increase meeting probability is to agree to not come early but wait moderately for each other. This can help them avoid a Nash equilibrium of the subsection with low meeting probability.

2 Literature Review

A game similar to my rendezvous game is the battle of the sexes game. Described by Luce and Raiffa (1989), this game has the two players, who want to go to the same place as the other player. However, each player has a different preference on where they should meet. This means, just like in the rendezvous game, players have incentives to both cooperate with and exploit each other. While the players want to agree on the destination, they want to agree on the destination favorable to them and not the other player.[1]

Hausken (2005) considers a repeated battle of the sexes game where only one player cares about the future. In this game, caring about the future makes the player willing to risk conflict with the other player today. This is analogous to the result in my model that when players value the meeting more highly, meeting chance can decrease as players are more willing to tolerate miscoordinations and resulting meeting failures. Zapata et al. (2018) finds equilibria for the battle of sexes game under the assumption that the two players care about the utility of the other. This paper shows that when one of the players is pro-social, the mixed strategy Nash equilibrium where the meeting chance is between 0 and 1 are decimated. For my model, I show how a high value of meeting for one of the players can cause Nash equilibria where the meeting chance is between 0 and 1 to disappear.

My paper is also related to the study of R&D using hazard rates. In my model, I find the optimal decisions for players, whether they should come to the meeting and how long they should wait before abandoning the meeting place. To do this, I compare the hazard rate of the other

1. Farrell and Saloner (1985) analyzes a similar game with N firms that want to move to a better standard in the presence of network externalities. Farrell and Saloner (1988) also discusses a similar setting. However, in this later paper, they compare Nash equilibrium outcomes for models with communication and without communication and show that communication is more likely to result in coordination.

player's arrival and the cost of wait. In studying R&D, many papers have used the approach of finding the firms' optimal research decisions by comparing the hazard rate of invention and the cost of R&D. Kamien and Schwartz (1972) was the first to analyze multi-player R&D models using hazard rates of inventions. However, in this paper, the firm considered only the hazard rate of invention for the composite rival and not its own hazard rate of invention. By doing so, the firm found the optimal invention time. In other words, the firm, unlike its rivals, is able to determine a invention time for its product.

All other subsequent papers I mention that study R&D using hazard rates instead have hazard rates of invention for all firms and find game theoretic solutions by considering the firms' hazard rate with the costs of R&D. Loury (1979) and Lee and Wilde (1980) deal with a setting where every firm is identical. By comparing the hazard rate of invention and costs of R&D, firms find the optimal investment in R&D to maximize expected profits.[2] Reinganum (1983) analyzes an asymmetric setting with an incumbent firm and a challenger firm. This paper finds that the challenger invests more in R&D because the challenger has more to benefit from investment since it does not have current revenue. Doraszelski (2003) shows that when the firm's hazard rate of invention is a weakly increasing function of the firm's knowledge stock, the firm that is behind in R&D may invest more in R&D than the firm that is ahead.[3]

Many different causes can result in varying travel times (Kwon et al. 2011; Wong and Sussman 1973). Iida (1999) defines travel time reliability as the probability of reaching the destination within a given time. The value of travel time reliability depends on the traveller's preferences. Polak (1987) and Senna (1994) derived expected utility formulas in which the value of travel time reliability was made explicit. Small (1982) was the first to derive the Noland-Small equation.[4] The Noland-Small equation attempts to take into account the realistic considerations that go into scheduling a trip. Travellers want shorter travel times. They also do not want to arrive too early or too late. From the equation, I utilize the idea that the cost of travel time, cost of arriving early and the cost of arriving late can be separated and expressed additively. In the context of my model, the cost of arriving early becomes the cost of increased wait and the cost of arriving late becomes the loss from decreased meeting chance.

2. Choi (1991) is the seminal paper in which firms have the option to drop out from R&D. In my model, this dropping out is comparable to giving up on the meeting and abandoning the meeting place. Choi (1991) assumes that firms do not know their hazard rates of inventions. However, they observe the state of the other firm. Therefore, if the other firm makes partial progress on the invention, depending on the parameters, this can lead the firm to either drop out because of the technological gap or continue R&D because the firm now has reason to believe that the hazard rate of invention is high.

3. Bag and Dasgupta (1995), Malueg and Tsutsui (1997) and Moscarini and Squintani (2010) extended Choi (1991). In Bag and Dasgupta (1995), firms now have the option to either announce their partial progress or hide it. If the partial progress is made early, it is announced but otherwise it is hidden. In Malueg and Tsutsui (1997), the invention might be impossible. If no firm succeeds in inventing for long, firms might drop out from R&D. In my model, this is comparable to realizing that the meeting will never succeed because the other player will never come to the meeting place and abandoning the meeting place. In Moscarini and Squintani (2010) when a firm exits, the other firm infers that the invention may be too difficult. Therefore, even when firms are asymmetrical, the times when the firms drop out of R&D may be close to each other.

4. In Noland and Small (1995), the equation (cost function) was specified as

$$C = \alpha T + \beta(SDE) + \gamma(SDL) + \Theta D_L.$$

α, β, γ and Θ are parameters. T is travel time. SDE is how overmuch early the person arrived. SDE is how overmuch late the person arrived.

3 Model

The rendezvous game has two people, player 1 and player 2, who make decisions about the meeting. Each person needs to decide by herself 1) whether she wants to come to the meeting at all, 2) when to depart for the meeting and 3) how long she waits for the other person at the meeting place. In making these decisions, people consider both the consequences of their own actions and the actions of the other person. While there is a benefit to a successful meeting, this comes at a cost of travelling time and potential waiting time. Leaving too early for the meeting place can mean the person has to wait longer for the other person. Leaving too late might cause the person to miss the other person entirely. People take these factors into consideration while choosing when to leave for the meeting.

3.1 Payoffs

To model the considerations of the player $i \in \{1,2\}$, I use an expected utility framework following Morgenstern and Von Neumann (1953). When a player does not come to the meeting, her utility is 0. Otherwise, I can define the following utility function.

$$u_i(m_i, w_i, r_i) \equiv m_i - c_i(r_i, w_i) \tag{1}$$

The ex-post utility, u_i depends on m_i, w_i and r_i. $m_i \in \{0, \bar{m}_i\}$ is player i's benefit. Here $\bar{m}_i > 0$ is the benefit player i gets from a successful meeting. $m_i = \bar{m}_i$ if and only if the players meet. $w_i \in R_+$ is how long the player waited, her wait time. r_i represents the time spent to travel to the meeting place. c_i is player i's cost. It is weakly increasing in both r_i and w_i. Its codomain is R_+. From equation 1, we can see that the benefit of meeting, $\bar{m}_i$, and the cost function, c_i, determine how the players are affected by the material outcomes of the game.

3.2 Sources of variation

For a given rendezvous, many random variables alter the travel, wait and meeting of players. In my model, there are two sources of fundamental randomness for the rendezvous. The first is the variation in preparation. Depending on the occasion, people may get ready earlier or later for the trip than expected. For instance, they may get up early or later than usual from bed. I model this variation by assigning independent stochastic start time, s_i to each player i. The codomain of any s_i is R_+. A player's start time is defined as the earliest time that the player can depart for the meeting place. They represent when the game "starts" for each player in the sense that all their travel and wait decisions can only be effectuated after their start time.

Secondly, there is the variation in travel time. Wong and Sussman (1973) classifies the components in travel time variation into three categories. Simply put, some of the travel time variation is from predictable factors such as rush hours or planned road construction while some of it is from factors unpredictable beforehand such as traffic accidents. Intuitively, variation in preparation represents the "something came up before I left" scenario and variation in travel time represents the "something came up on the way" scenario. I assume that the predictable factors are common knowledge and given by the game since these factors would hold constant for a rendezvous and players are unlikely to have private information on road conditions beforehand. It is after they start travelling that players gain private information on their own travel times as the unpredictable factors come into play. Thus, when possible[5], we model r_1 and r_2, the travel

5. If player i does not come to the meeting place, r_i does not exist. player i must always come to the meeting for r_i to be a random variable.

times to the meeting place, as random variables the realization of which players do not know before travelling.

By "continuous rendezvous games", I mean that in this paper, for the most part, r_i follows a continuous distribution.[6] The codomains of the r_i's are R_+. The r_i's are independent of each other and the s_j's. If a CDF exists for r_i, the CDF is G_i and if the PDF exists for r_i, it is g_i.

3.3 Stages

This is a 2-stage sequential game. player i receives a private start time, s_i. Then, player i chooses whether to depart for the meeting place. If she chooses to depart, she also chooses a departure time, $d_i \geq s_i$ and receives an arrival time, $a_i = d_i + r_i$. d_i, r_i and a_i are also private. Given that player(s) i and j chooses (choose) to depart, r_i and d_j are conditionally independent. Later on, in specifying the distribution of a_i, $\Gamma_i(t) = P(a_i \leq t)$ is used. If player i always comes to the meeting, $\Gamma_i(t)$ is a CDF of a_i and $\Gamma_i(t) = \int G_i(t-d_i)P(dd_i)$.[7] If $\Gamma_i(t)$ has a PDF, it is written as $\gamma_i(t)$. The follwing is the specification of the stages, which is depicted in figure 1.

- Pre-game Setup
  1. Nature assigns each player i a random start time, $s_i \geq 0$.
- Simultaneous Actions in Stage 1
  1. Each player decides on whether they will travel to the meeting place.
  2. Each player i who decided to travel decide the time at which they will depart for the meeting place. This time is called the departure time or $d_i \geq s_i$.
- Simultaneous Actions in Stage 2
  1. Nature decides the r_i's for player who decided to travel in stage 1.
  2. After seeing their own $a_i = d_i + r_i$'s, each player i who decided to travel privately decides the time beyond which they will not wait and instead, abandon the meeting place. This time is called planned abandonment time or ζ_i. player i who travels chooses $\zeta_i \in [0, \infty]$, in other words, ζ_i is an element of the extended real line.
- Payoffs
  1. Players' payoffs are their expected utilities from the game. Given all the decisions of the two stages, the rendezvous game is played out in the following way. Players who decided not to come do nothing. Players who decided to come depart for the meeting place at d_i and realize travel time, r_i. Now, their arrival time is $a_i \equiv d_i + r_i$. Given their arrival time, we also have their actionable abandonment time, $z_i \equiv max\{a_i, \zeta_i\}$. This z_i is private information. The rendezvous is successful if and only if both players come and $max\{a_1, a_2\} \leq min\{z_1, z_2\}$. If the rendezvous fails, players who came leave the meeting space at z_i.

6. It takes on uncountably many values.

7. I will explain the integral, $\Gamma_i(t) = \int G_i(t-d_i)P(dd_i)$. If player i is to arrive no later than t, given d_i, player i's travel time must be no more than $t - d_i$. Hence, the integrand is $G_i(t-d_i)$. I integrate over all d_i.

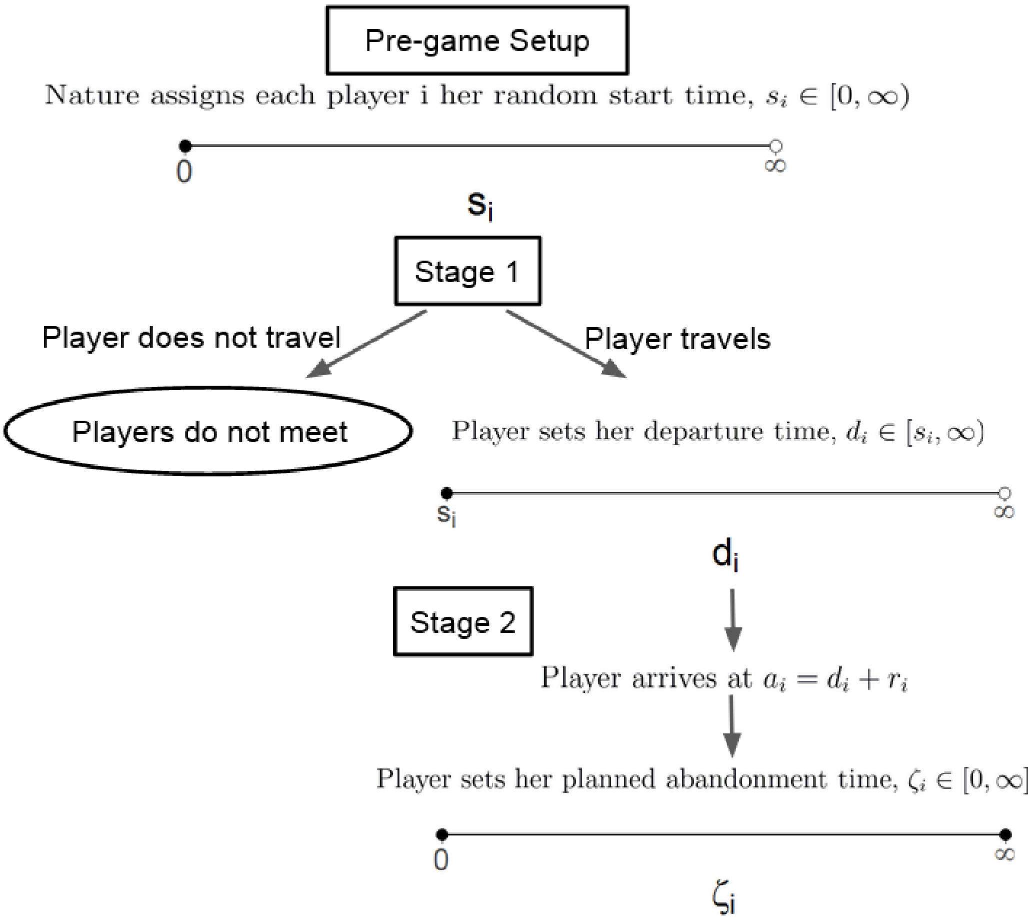

Figure 1: The stages of the game

For convenience, I define a random variable M the following way.

$$M = \begin{cases} 1 & \text{if } max\{a_1, a_2\} \leq min\{z_1, z_2\} \text{ (i.e. if rendezvous succeeds)} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By this definition, E(M) becomes the probability of the players meeting.[8]

A noteworthy point is that the players do not decide on their wait times, the w_i's directly. In fact, players indirectly plan their wait times using their planned abandonment time. A player's actual wait time depends on when she and the other player arrive. The following is the exact formula for wait times.

$$w_i = \begin{cases} max\{a_1, a_2\} - a_i & \text{if M=1} \\ z_i - a_i & \text{otherwise} \end{cases} \tag{3}$$

The logic for this indirection is similar to before. Once the players depart for the meeting place, there is nothing they can do to change the other player's arrival time. Furthermore, how long the players wait or when the players abandon the meeting place depends on the probability

8. By Lebesgue's dominated convergence theorem, M is Lebesgue integrable and equivalently, $E(M)$ is finite.

distribution of the other player's arrival. Given the player's departure and arrival time, the ex ante distribution of the other player's arrival tells the player when it is no longer worth it to wait for the other player. The player would set that time as the planned abandonment time.[9]

Actionable abandon time, z_i, exists to deal with cases where a player arrives after her planned abandonment time. In that case, the player would want to leave immediately unless her opponent is already at the meeting place. Then, the arrival time, not the planned abandonment time is when she abandons the meeting place should she fail to meet. The meeting happens if and only if both players arrive before any player would abandon the meeting place,

In this game, players can play mixed strategies. Thus, for a given arrival time, a_i, player i might have infinitely many optimal ζ_i's or a unique optimal ζ_i. Of course, the same is true for z_i as well. Therefore, I need notations that can signify those different cases. The following introduces those notations.

I can define the correspondence $\zeta_i^*(a_i)$ the following way when the set on the right-hand side is not empty for a given arrival time, a_i.

$$\zeta_i^*(a_i) = \{\zeta^* | \forall d_i \text{ and } \zeta_i, E(u_i|\zeta^*, d_i, a_i) \geq E(u_i|\zeta_i, d_i, a_i) \text{ when both sides of } \geq \text{ exist}\}$$

Suppose given an arrival time, a_i and $\zeta^* \in R^+$, $E(u_i|\zeta^*, d_i, a_i) > E(u_i|\zeta, d_i, a_i)$ for any d_i and $\zeta \neq \zeta^*$. Then, I define the function, $\zeta_i^*(a_i) = \zeta^*$. The correspondence and the function of $\zeta_i^*(a_i)$ are used to show what the optimal planned wait time(s) are for a given arrival time, a_i. $z_i^*(a_i)$ is defined in the same way as $\zeta_i^*(a_i)$.

3.4 Model Analysis

Suppose player i's expected benefit and cost are decreasing in her departure time and that they are increasing in her planned abandonment time. Then these features inform the players about the costs and benefits they have to consider when setting their departure times and planned abandonment times. Informally speaking, waiting longer has the benefit of making the player more likely to meet the other player but also has the cost of increased wait time. Similarly, departing later has the benefit of reducing the player's wait but has the cost of reducing the meeting chance.

Given these costs and benefits, I will often use the first order conditions for maxima to solve the model. I will find the optimal planned abandonment time or the actionable abandonment time (Recall that $z_i \equiv max\{a_i, \zeta_i\}$.) by comparing the marginal benefit derived from increased meeting chance with the marginal cost derived from increased wait. Similarly, I will find the optimal departure time by comparing the marginal benefit derived from decreased wait with the marginal cost derived from decreased meeting chance.

Under some conditions (which are fully stated in proposition 6 in appendix 1), player i's conditional expected utility, $\frac{\partial E(u_i|d_i,a_i,z_i)}{\partial z_i}$ has the following derivative with respect to z_i.

$$\frac{\partial E(u_i|d_i, a_i, z_i)}{\partial z_i} = \gamma_{-i}(z_i)\bar{m}_i - (E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i) + 1 - \Gamma_{-i}(z_i))\frac{\partial c_i(a_i - d_i, z_i - a_i)}{\partial z_i} \tag{4}$$

The above equation states that the marginal benefit of actionable desertion time is the player's value of the meeting times the marginal probability of the other player's arrival at the actionable abandonment time. (To be precise, the marginal probability of the other player's

9. On the contrary, if the players were to set their wait times directly and their planned abandonment times indirectly before travelling, they would be unable to abandon the meeting place or wait optimally because the variation in travel time to the meeting place would affect when they actually abandon the meeting place.

arrival is actually the value of the PDF of the other player's arrival). On the other hand, the cost of actionable desertion time is the conditional expectation the players haven't met, $E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i)+1-\Gamma_{-i}(z_i)$ times the marginal cost of waiting at the actionable desertion time. To state intuitively, in deciding whether to wait marginally more, the player considers the benefit given by multiplying the value of the meeting and the probability that the other player player will arrive during the marginal wait time. The player considers the cost given by multiplying the probability that the player actually has to wait and the marginal cost of wait.

I will explain this "probability" that the player actually has to wait in more detail. Obviously, the player only needs to wait if she hasn't met the other player yet. If she has, there is no wait. When the player has not meet the other player, she considers the two potential possibilities for why this has happened. The other player may have left early or he may have not come yet. To be elaborate, the first possibility, $E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i)$ is the conditional expectation that the other player already came and left the meeting place. The second possibility, $1-\Gamma_{-i}(z_i)$ is the probability that the other player will arrive in the future.

When player i's arrival time, a_i is known and the probability that player -i abandoned the meeting place before this arrival time, $E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i)$ is 0, equation 4 can be restated as follows.

$$\frac{\partial E(u_i|d_i,a_i,z_i)}{\partial z_i}=\gamma_{-i}(z_i)\bar{m}_i-(1-\Gamma_{-i}(z_i))\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i} \tag{5}$$

Because equations 4 and 5 are difficult to analyze, I use the following formula. This formula will be the main method in the text of this paper to explain and graph the marginal benefit and cost of actionable abandonment time. When $E(M|a_i,z_i)<1$ is also true, by proposition 6's (2), the sign of $\frac{\partial E(u_i|d_i,a_i,z_i)}{\partial z_i}$ can be known the following way.

$$\begin{aligned}&\frac{\partial E(u_i|d_i,a_i,z_i)}{\partial z_i}\lesseqgtr 0\\&\leftrightarrow\\&\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}\bar{m}_i-\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}\lesseqgtr 0\end{aligned} \tag{6}$$

In this case, instead of considering the sign of the marginal utility directly, player i can consider the sign of $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}\bar{m}_i-\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}$ instead and gets the same result. In this context, $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}\bar{m}_i$ is the marginal benefit of actionable abandonment time and $\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}$ is the marginal cost of actionable abandonment time. $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}$ is the hazard rate of player -i's arrival at z_i. This hazard rate represents the marginal probability that the other player will arrive during the marginal wait time given that she has not arrived yet. As before, $\bar{m}_i$ is player i's value of the meeting and $\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}$ is the marginal cost of actionable abandonment time.

In other words, instead of looking at the marginal utility of actionable abandonment time directly, player i can use the fact that the wait time only matters when the other player has not arrived yet. The player can assume that the other player has not arrived yet. Given this assumption, in deciding whether to wait more marginally, the player can weigh her value of how likely the other player is likely to arrive if the player waits marginally more against the marginal cost of wait.

4 Results

4.1 Degenerate start time, s_i

Assumption 1. *The following formulas hold for all $i \in \{1,2\}$.*

$$s_i = 0$$

If r_i exists, $r_i \sim U(0,1)$.

$$c_i(r_i, w_i) = r_i + w_i$$

Assumption 1 specifies the start times, the s_i's, the travel times, the r_i's and the costs, the $c_i(r_i, w_i)$'s for this subsection. Here, players always start at time 0 and their travel time is distributed uniformly. Cost is the sum of travel time and wait time, w_i.

Assumption 2. *Suppose that for any player i, fixed $a_i \geq 0$ and fixed $\zeta_i \geq a_i$, $E(M|a_i, \zeta_i) = 1$. Then, for any ζ that player i plays for a given a_i, $\zeta \leq \zeta_i$.*

Assumption 2 caps how high planned wait time and actionable wait time can be for its cases. It states that for a given arrival time of $a_i \geq 0$, if waiting till time $\zeta_i \geq a_i$ is sufficient to guarantee a meeting probability of 1, player i never waits beyond time ζ_i. In other words, given arrival times, players do not set actionable wait times that are so high that the actionable wait times are beyond what it is necessary for them to always meet the other person.[10]

Now, I will introduce a noteworthy lemma used in proving the necessary and sufficient condition for pure strategy Nash equilibria with positive meeting chance under assumption 1. This necessary and sufficient condition is loosely stated in proposition 1 which I will show in this subsection. It is rigorously stated in proposition 7. Appendix 2 contains proposition 7 and all proofs not found here.

Lemma 1. *Under assumption 1, suppose that $\bar{m}_i < 0.5$ for some i. There is no Nash equilibrium with $E(M) > 0$.*

Proof.

$$E(r_i) = 0.5$$

$$E(c_i) = E(r_i) + E(w_i) \geq 0.5$$

Thus if $\bar{m}_i < 0.5$, player i prefers to not come to the meeting place. If player i has a 0 probability of coming to the meeting place, player -i prefers to not come as well. ■

The above lemma establishes a lower bound of 0.5 on the values of meeting for both players in a a Nash equilibria with positive meeting chance. The fact that both players are willing to come to the meeting means that their values of meeting is at least as great as their expected travelling costs, 0.5. For the special case where a player's value of meeting is exactly 0.5, it is easy to find the properties of the pure strategy Nash equilibria. For all other cases, we can solve under $\bar{m}_i > 0.5$ for all i.

10. In cases where the players always meet, there is no drawback to increasing planned abandonment time as doing so will not actually increase the player's expected wait. Imposing assumption 2 serves to rule out inconsequential equilibria where players' planned wait times are too high to happen.

Proposition 1. *Under assumptions 1 and 2, the following for some i is necessary and sufficient for a pure strategy Nash equilibrium with $E(M) > 0$. (In stating the following, I ignore 0 probability events and planned abandonment times for cases where the player has a 0 probability to wait)*

(1) $\bar{m}_i \geq max\{\frac{(d_i-d_{-i})^2+1}{2(d_i+1-d_{-i})}, \frac{1}{2}+\frac{(d_i+1-d_{-i})^3}{6}+d_{-i}-d_i\}$

(2) $\bar{m}_{-i} \geq \frac{1}{2}+\frac{(d_i+1-d_{-i})^3}{6}$

(3) Players play d_1 and d_2 such that $d_i \leq d_{-i} < d_i+1$.

(4) Any player j plays $\zeta_j = d_{-j}+1$.

The above proposition is the key result of this subsection. It establishes the necessary and sufficient condition for pure strategy Nash equilibria with positive meeting probability under assumptions 1 and 2. This condition is stated in (1)∼(4) and its most notable result is (4). (4) means that in the Nash equilibria, players always way till the other player comes, ergo the meeting probability is 1. Using symmetry, I explain these Nash equilibria for the $i = 2$ case where $d_2 \leq d_1$ by (3). In the proposition, (1) and (2) establishes the lower bounds on the values of meeting for players 2 and 1, respectively. (3) specifies that this is a Nash equilibria where player 2 departs earlier than player 1 or both players depart simultaneously. It also specifies that player 2 does not depart so early that player 1's earliest arrival time is equal to or later than player 2's latest arrival time.[11] (4) establishes the planned abandonment times for the players in terms of the other player's departure time.

Since the proof of proposition 1 is complicated, here I will only describe the rough flow of logic for it. Lemma 1 establishes that I can solve for the case where players value the meeting more than 0.5. Given this, I establish that in a Nash equilibrium, there is at least a minimal wait. Based this minimal wait, strategic complimentary of waits takes effect and gives the result that players always wait till the other player comes in a Nash equilibrium. The strategic complimentary of waits means that if the other player waits for a player, the player is also likely to wait for her because the player knows that the other player has not abandoned the meeting and will come in the future. On the other hand, if the other player does not wait for the player, if the player knows her wait might be futile because the other player may have already left. So the player will likely not wait in this case.

I will explain (1) and (2) from the proposition using the following definition.

Definition 1.

$$m_2'(d_1,d_2) \equiv \frac{1}{2}+\frac{(d_2+1-d_1)^3}{6}+d_1-d_2$$

$$m_2''(d_1,d_2) \equiv \frac{(d_2-d_1)^2+1}{2(d_2+1-d_1)}$$

$$m_1'(d_1,d_2) \equiv \frac{1}{2}+\frac{(d_2+1-d_1)^3}{6}$$

11. Note that in lemma 16 of appendix 2, the condition was $d_{-i} \leq d_i+1$ instead of $d_1 < d_2+1$. Here, I have the additional requirement that $d_1 = d_2+1$ cannot be true. The addition of (1) means that no pure strategy Nash equilibrium with $d_1 = d_2+1$ exists as it would require an infinitely high value of meeting for player 2.

Note that by (1) and (2), $\bar{m}_2$ needs to satisfy both $\bar{m}_2 \geq m''_2(d_1,d_2)$ and $\bar{m}_2 \geq m'_2(d_1,d_2)$ while $\bar{m}_1$ only needs to satisfy $\bar{m}_1 \geq m'_1(d_1,d_2)$. $\bar{m}_2 \geq m''_2(d_1,d_2)$ comes from the requirement that player 2 weakly prefers not to delay departure. (For player 1, the condition that she weakly prefers to not delay departure is not binding.) $\bar{m}_2 \geq m'_2(d_1,d_2)$ and $\bar{m}_1 \geq m'_1(d_1,d_2)$ come for the requirement that player 2 and player 1 respectively weakly prefer to come to the meeting place. There is no upper bound on the players' values of meeting. Once the lower bounds on the players' values of meeting in (1) and (2) are met, players can have much higher values of meeting. Given $d_1 - d_2$, either player can value the meeting more highly in a pure strategy Nash equilibrium.

Since players always meet in the Nash equilibria of the proposition, $\bar{m}_2$ and $\bar{m}_1$ are respectively player 2 and 1's expected benefits in the Nash equilibria. $m'_2(d_1,d_2)$ and $m'_1(d_1,d_2)$ are respectively player 2 and 1's expected costs in the pure strategy Nash equilibria. Note that when $d_1 = d_2$, the expected costs are equal and $\bar{m}_2 \geq m''_2(d_1,d_2)$ is not binding. Proposition 1's (3) says $d_2 \leq d_1 < d_2 + 1$. There is no pure strategy Nash equilibrium with $d_1 - d_2 = 1$. This is because $d_1 - d_2 \to 1$, $m''_2(d_1,d_2) = \frac{(d_2-d_1)^2+1}{2(d_2+1-d_1)} \to \infty$.

Proposition 2. *When $d_1 - 1 < d_2 \leq d_1$, the following holds.*

(1) m'_2, m''_2 and $m'_1 + m'_2$ are increasing in $d_1 - d_2$.

(2) m'_1 is decreasing in $d_1 - d_2$.

The above proposition helps analyze m'_1, m'_2 and m''_2. In the Nash equilibria of proposition 1, as $d_1 - d_2$, the distance between the departure times of the players increases, m'_2, the expected cost of player 2, the player who departs weakly early increases and m'_1, the expected cost of player 1, the player who departs weakly early decreases. Since m''_2 also increases as the distance between the departure times of the players increases, the comparatively earlier a player arrives, the higher her required value of the meeting for a Nash equilibrium of proposition 1. (Recall that the players always meet in the Nash equilibria of proposition 1.)

$m'_1 + m'_2$ is increasing in $d_1 - d_2$. Therefore, the sum of players' expected costs is increasing in the difference of players' departure times. Players' expected utilities are decreasing in the difference of players' departure times. Thus, of the Nash equilibria of proposition 1, the symmetric equilibria maximizes social welfare and the more asymmetric the equilibria is in terms of departure times, the smaller the social welfare.

(4) means that in the Nash equilibria, any player j has a constant planned abandonment time, ζ_j. This is possible because the cost defined by assumption 1 is the sum of travel time and wait time. Because of this, once players arrive to the meeting place, they can treat their travel time as sunk cost. Also the players know that the probability that the other player has already left the meeting place is 0. If a player is still waiting for the other player at time t, for her, it does not matter when she arrived, how long it took her to get there, or how much she waited so far. Therefore, regardless of those circumstances, the player can set a single planned abandonment time for time t. Using this logic, if the player who arrives immediately finds it optimal to set her planned abandonment time to t', she will feel the same way even if she arrives later.

Example 1 is a specific case of the Nash equilibria of proposition 1. In example 1, the constraints on $\bar{m}_1$ and $\bar{m}_2$ are binding.

Example 1. *Under assumption 1, there exists a pure strategy Nash equilibrium characterized by the following.*

(1) $\bar{m}_2 = max\{\frac{(d_2-d_1)^2+1}{2(d_2+1-d_1)}, \frac{1}{2} + \frac{(d_2+1-d_1)^3}{6} + d_1 - d_2\} = \frac{(d_2-d_1)^2+1}{2(d_2+1-d_1)} = 1.25$

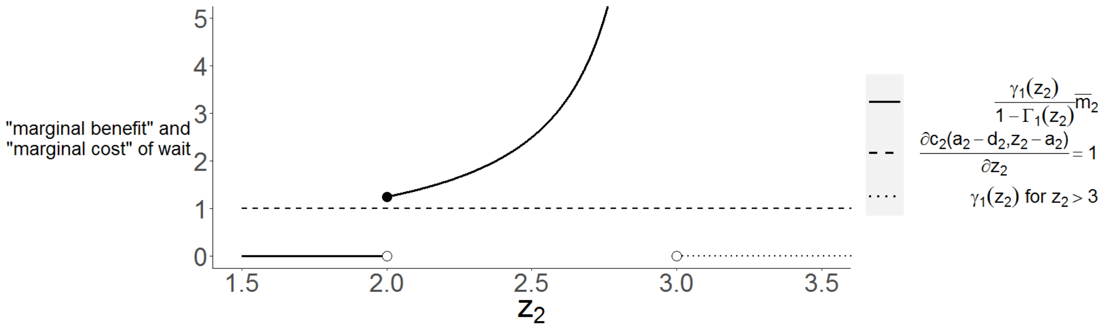

Figure 2: $\frac{\gamma_1(z_2)}{1-\Gamma_1(z_2)}\bar{m}_2$, $\gamma_1(z_2)$ and $\frac{\partial c_2}{\partial z_2}$ when $a_2 = 1.5$ for example 1

(2) $\bar{m}_1 = \frac{1}{2} + \frac{(d_2+1-d_1)^3}{6} \approx 0.52$

(3) $d_1 = 2$

(4) $d_2 = d_1 - 0.5 = 1.5$

(5) $\zeta_1 = d_2 + 1 = 2.5$

(6) $\zeta_2 = d_1 + 1 = 3$

Proof. The proof is by proposition 7 in appendix 2. ■

Now using hazard rate analysis, I will roughly explain why for specific arrival times, players find it optimal to wait till the other player arrives. For this, I use example 1 and figure 2 which is on this example. However, the explanation applies to any player in any Nash equilibrium of proposition 1. The figure draws functions with z_i, the actionable abandonment times on the X-axis. This will help me find the optimal z_i. Figure 2 depicts $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}\bar{m}_i$, player –i's hazard rate of arrival at z_i times player i's value of meeting, $\bar{m}_i$, player –i's density of arrival at z_i, $\gamma_{-i}(z_i)$ and finally $\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}$, the marginal cost of wait. I will consider $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}\bar{m}_i$ to be the marginal benefit of wait. The figure is drawn using proposition 8 in appendix 2.

The figure illustrates player 2 when she arrives at $a_2 = 1.5$. However, the reason that any player waits till the other player arrives is similar for any other arrival time of the player that is possible in the example. In this analysis, I can apply the hazard rate analysis of formula 6 because the probability that player 1 already arrived and left is 0.

The figure shows that no actionable abandonment time, z_2 between 1.5 and 3 is optimal. When z_2 is between 1.5 and 2, the marginal benefit of wait is less than the marginal cost of wait. So this is not optimal. When z_2 is 2 or greater but less than 3, the marginal benefit of wait is greater than the marginal cost of wait. So this is not optimal either. The figure also shows that player 2 does not prefer any actionable abandonment time, $z_2 > 3$ to $z_2 = 3$. This is because $\gamma_1(z_2) = 0$ here and player 1 never arrives after 3 which makes waiting after 3 futile. Therefore, the remaining candidates are $z_2 = 1.5$ and $z_2 = 3$. If player 2 sets $z_2 = 1.5$, she never meets player 1. Since player 2 has a high value of the meeting at $\bar{m} = 1.25$, she prefers to set $z_2 = 3$ and wait till the other player arrives which lets her always meet player 1. $z_2 = 3$ is optimal.

4.2 Uniformly distributed start time, s_i

This subsection deals with the model when both players' start times, s_i's are uniformly distributed. Recall that a player departure times d_i must be later than or equal to her s_i. Therefore, start time variation means that players may be unable to depart as early as they want to. The travel times, r_i's are also uniformly distributed for players who travel. For this subsection, proofs not found here are in appendix 3. The exact distributions are specified in the following assumption. This assumption for both players lays out the basic setting of the model.

Assumption 3.
$s_i \sim U(0,1)$
If r_i exists, $r_i \sim U(0,1)$.

When players are able to depart as early as they want to because of start time variation, they might depart later than they want to or not depart for the meeting. In order to describe these phenomena and strategies, I define two additional variables, $\underline{s}$ and $\bar{s}$ in the definition below. The main focus of this subsections is symmetric Nash equilibria when the players face such constraints.[12]

Definition 2. $\underline{s} \in [0,1)$ *is used for the earliest departure time by the players' strategies.*
$\bar{s} \in (\underline{s},1]$ *is used for the earliest departure time by the players' strategies.*

In the following assumption I explain how exactly player's strategies depend on $\underline{s}$ and $\bar{s}$.

Assumption 4.
If $s_i \leq \underline{s}$, $d_i = \underline{s}$.
If $s_i \in (\underline{s}, \bar{s}]$, $d_i = s_i$.
If $s_i > \bar{s}$, player i does not depart for the meeting place.
If player i departs for the meeting place, $\zeta_i = \underline{s} + 1$.

The above assumption uses $\underline{s}$ and $\bar{s}$ to specify player i's pure strategy in the symmetric pure strategy Nash equilibria that I will explicate in this subsection. By the assumption, a player's strategy in these pure strategy Nash equilibria is this. If the player starts before $\underline{s}$ or at $\underline{s}$, player departs at $\underline{s}$. This behavior can be thought as the player waiting for a more suitable departure time if she starts too early and explains why $\underline{s}$ is the earliest departure time players may choose. If the player starts after $\underline{s}$ but not after $\bar{s}$, player departs immediately. This represents a person realizing that she started later than she hoped to and departing immediately since she believes that it is not too late. If the player starts after $\bar{s}$, the player does not depart for the meeting. This can be explained by the player realizing that it is too late to go to the meeting now.

Therefore, these Nash equilibria realistically represent how people may find themselves ready too early or too late to travel to a meeting and how people make departure decisions based on the arising time considerations. When a player departs for the meeting place, her planned abandonment time, ζ_i is always $\underline{s}+1$. This means that by deciding the players' earliest arrival time, $\underline{s}$ in the Nash equilibrium, I also decide the time till which they will wait at the latest, $\underline{s}+1$.

Next, for analyzing the players' arrival time distributions, I will define a CDF and a PDF based on assumptions 3 and 4. The actual values for $P(a_i \leq x)$ and $\frac{\partial P(a_i \leq x)}{\partial x}$ based on these assumptions are found in lemma 4 in appendix 3. Unfortunately, these $P(a_i \leq x)$ and $\frac{\partial P(a_i \leq x)}{\partial x}$ are not necessarily a CDF nor a PDF. This is because player i does not come to the meeting place when she starts after $\bar{s}$. This means a_i may not be a random variable. Therefore, in order to make a CDF and a PDF from the players' arrivals, I insert fake arrivals for $[10,11]$ while leaving other arrivals the same. This is seen in the following definition of the CDF and the PDF.

12. Readers may ask why the solution concept is not Bayesian Nash equilibrium. This is because there is no communication in stage 1 and the game ends in stage 2. Players do not receive any signals based on which they can form beliefs and adjust their actions.

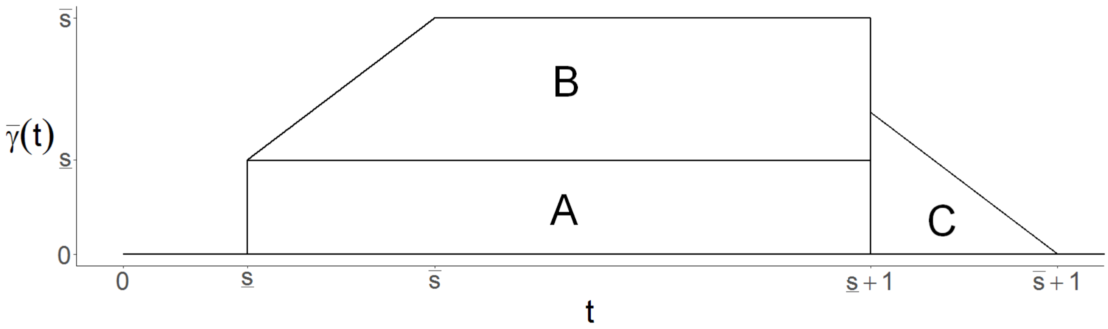

Figure 3: A PDF of definition 3 with polygons delineated

Definition 3.

(1) The following is a CDF.

$$\bar{\Gamma}(x) = \begin{cases} 0 & x \leq \underline{s} \\ \frac{x^2 - \underline{s}^2}{2} & x \in [\underline{s}, \bar{s}] \\ \bar{s}x - \frac{\bar{s}^2 + \underline{s}^2}{2} & x \in [\bar{s}, \underline{s}+1] \\ \bar{s} - \frac{(\bar{s}+1-x)^2}{2} & x \in [\underline{s}+1, \bar{s}+1] \\ \bar{s} & x \in [\bar{s}+1, 10] \\ (1-\bar{s})x + 11\bar{s} - 10 & x \in [10, 11] \\ 1 & x \geq 11 \end{cases}$$

(2) The following is a PDF of $\bar{\Gamma}(x)$.

$$\bar{\gamma}(x) = \begin{cases} 0 & x < \underline{s} \\ x & x \in [\underline{s}, \bar{s}] \\ \bar{s} & x \in [\bar{s}, \underline{s}+1] \\ \bar{s}+1-x & x \in (\underline{s}+1, \bar{s}+1] \\ 0 & x \in [\bar{s}+1, 10) \\ 1-\bar{s} & x \in [10, 11] \\ 0 & x > 11 \end{cases}$$

The CDF and PDF from the above definition lets me consider the players' arrival times as random variables specified by a CDF and a PDF. This is useful for understanding and proofs. I can understand the CDF and the PDF as those that accurately describe the players' arrival times in $[0,8]$. In the actual games that follow assumptions 3 and 4, since player i never arrives after time 2, Player -i has no incentive to wait after time 2. Proving ζ_{-i} is optimal for player -i when ζ_{-i} is restricted to satisfy $\zeta_{-i} \in [0,8]$ and player i's arrival time follows definition 4 and $\zeta_i = \underline{s}+1$ is equivalent to proving that the same ζ_{-i} is optimal for player -i when player i's strategy follows assumptions 3 and 4.

From definition 3, the PDF, $\bar{\gamma}$ is depicted in figure 3. In the figure, I can see that the probability of arriving after $\underline{s}+1$ is comparatively lower than the probability of arriving before $\underline{s}+1$ or at $\underline{s}+1$. For the A rectangle, the vertical edges are from 0 to $\underline{s}$. The horizontal edges are from $\underline{s}$ to $\underline{s}+1$. This rectangle is from a player's departure at $\underline{s}$. Under assumption 4, a player can only have a positive probability of departing at $\underline{s}$ and she has a 0 probability of departing at any other point. The area of A is $\underline{s}$. The higher the $\underline{s}$, the greater its area.

Going back to figure 3, the B quadrilateral is from the player's departure after $\underline{s}$ for cases where the player arrives before $\underline{s}+1$ or at $\underline{s}+1$. The upward sloping edge of the B quadrilateral is due to the fact that if the player starts after $\underline{s}$ but not after $\bar{s}$, she departs immediately, adding on to the area of the B quadrilateral. Lastly, the C triangle is from cases where the player arrives after $\underline{s}+1$. Player only arrives after $\underline{s}+1$ when she departs after $\underline{s}$. This results in the downward sloping edge of the C triangle, which shows how the density of the players' arrival decreases after $\underline{s}+1$. In fact, unless $\underline{s}=0$, the PDF jumps downwards at $\underline{s}+1$. This decline in the PDF justifies why the players set their planned abandonment times to $\underline{s}+1$ and do not wait after $\underline{s}+1$.

The following definition introduces two functions used in concisely stating and proving the results of this subsection.

Definition 4.

$$\bar{i}(\underline{s},\bar{s}) \equiv \frac{6+2(\bar{s}+3)(\underline{s}+1-\bar{s})^3+3((\bar{s}-\underline{s})^2-2\underline{s})(\underline{s}+1-\bar{s})^2}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})}$$

$$\bar{w}(\underline{s},\bar{s}) \equiv \frac{1-\bar{s}}{\bar{s}-\underline{s}}+\frac{\bar{s}-\underline{s}}{2}$$

Proposition 3. *Under assumption 3 for both players, assumption 4 for both players is a pure strategy Nash equilibrium if and only if* $\bar{m}_1=\bar{m}_2=\bar{i}(\underline{s},\bar{s})\leq\bar{w}(\underline{s},\bar{s})$

$\bar{i}(\underline{s},\bar{s})$ is the function used for the indifference condition, $\bar{m}_i=\bar{i}(\underline{s},\bar{s})$. $\bar{w}(\underline{s},\bar{s})$ is the function used for the wait cap condition, $\bar{m}_i\leq\bar{w}(\underline{s},\bar{s})$. These two conditions are used to describe the symmetric pure strategy Nash equilibria of proposition 3. Under assumption 3 for both players, if and only if both conditions hold for both players, assumption 4 for both players is a Nash equilibrium. (In these Nash equilibria, by lemma 2 and formula 8, the meeting probability is $(\bar{s}-\frac{(\bar{s}-\underline{s})^2}{2})^2$.)

The first condition, $\bar{m}_i=\bar{i}(\underline{s},\bar{s})$ means that player i's utility must be 0 when she departs at $\bar{s}$ and has a planned abandonment time of $\zeta_i=\underline{s}+1$.[13] In other words, player i must be indifferent between departing at $\bar{s}$ to play $\zeta_i=\underline{s}+1$ and not departing at all. Therefore, I call this the indifference condition. In figure 4, $E(m_i|d_i)$ and $E(c_i|d_i)$ respectively represent player i's benefit and cost when she departs at d_i and plays $\zeta_i=\underline{s}+1$. (Figure 4 is drawn using proposition 9 in appendix 3.) In figure 4 and any Nash equilibrium of proposition 3, the two curves intersect at $d_i=\bar{s}$. Therefore, player i's expected utility at $d_i=\bar{s}$ is 0. So player i finds it optimal to come to the meeting when she starts before $\bar{s}$ or at $\bar{s}$. It also means that she finds it optimal to not come to the meeting if she starts later. If $\bar{m}_i$ is higher, $E(m_i|d_i)$ increases at $d_i=\bar{s}$ and the intersection moves to the right. In this case, player i prefers to increase her latest departure time. If $\bar{m}_i$ is lower, $E(m_i|d_i)$ decreases at $d_i=\bar{s}$ and the intersection moves to the left. In this case, player i prefers to decrease her latest departure time.

The second condition, $\bar{m}_i\leq\bar{w}(\underline{s},\bar{s})$ is necessary for a player i who arrives to weakly prefer a planned wait time of $\underline{s}+1$ to a greater one. Hence, I call this the wait cap condition. I will explain this condition roughly using figure 5. Figure 5 applies the aforementioned technique of converting the distribution of player -i's arrival time, a_{-i} to follow definition 3 so that a CDF and a PDF exist to represent a_{-i}. Then, I can perform hazard rate analysis under the restriction of $z_i\in[a_i,8]$.

13. This condition also implies that a participating player i weakly prefers a planned wait time of $\zeta_i=\underline{s}+1$ to a smaller one. This implication is shown by lemma 28 in appendix 3.

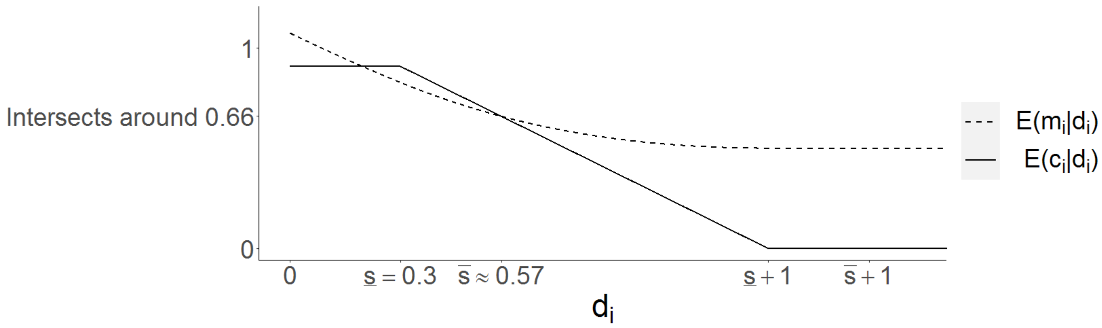

Figure 4: $E(m_i|d_i)$ and $E(c_i|d_i)$ when $\underline{s} = 0.3$ and $\bar{s} \approx 0.57$

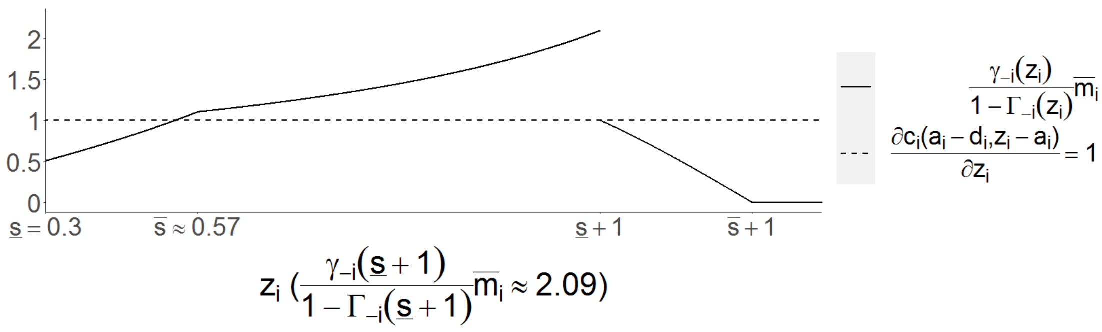

Figure 5: Hazard rate analysis using converted a_{-i} when $\underline{s} = 0.3$ and $\bar{s} \approx 0.57$

After the conversion, in figure 5, for $a_i = \underline{s}$, I draw $\frac{\gamma_i(z_i)}{1-\Gamma_i(z_i)}\bar{m}_i$, the hazard rate of player -i's arrival multiplied by player i's value of the meeting and $\frac{\partial c_i(a_i-d_i,z_i-d_i)}{\partial z_i} = 1$, the marginal cost of wait. (For this, I use results from example 2 in appendix 3.) $\frac{\gamma_i(z_i)}{1-\Gamma_i(z_i)}\bar{m}_i$ is considered the marginal benefit of wait. The x-axis is the actionable desertion time for player i, z_i. In this case, the probability that player -i arrived first and abandoned the meeting place before player i arrived is 0. Therefore, I can apply the hazard rate analysis of formula 6 by comparing the marginal benefit and the cost of wait.

I will explain why $z_i = \underline{s}+1$ is optimal when player i arrives at $a_i = \underline{s}$. A similar logic establishes that $\zeta_i = \underline{s}+1$ is optimal as a general strategy. The figure shows that any actionable abandonment time, $z_i < \underline{s}+1$ is not optimal. When $z_i < \underline{s}+1$, at first, marginal benefit less than marginal cost. Later, when marginal benefit is greater than or equal to marginal cost, player i prefers a greater z_i. So neither is optimal. Also, any actionable abandonment time, $z_i > \underline{s}+1$ is not optimal because marginal benefit is smaller than marginal cost. Therefore, the only remaining point, $z_i = \underline{s}+1$ is optimal.

For the figure, the wait cap condition, $\bar{m}_i \leq \bar{w}(\underline{s}, \bar{s})$ is binding. Because the condition is binding, when $z_i = \underline{s}+1$, the marginal benefit curve on the right "just touches" the marginal cost ray, $\frac{\partial c_i(a_i-d_i,z_i-d_i)}{\partial z_i} = 1$. (In other words, the right limit of $\frac{\gamma_i(z_i)}{1-\Gamma_i(z_i)}\bar{m}_i$ at $\underline{s}+1$ is 1.) Lower values of $\bar{m}_i$ will shift down the downward sloping part of the marginal benefit and higher values of $\bar{m}_i$ will shift up the downward sloping part of the marginal benefit. For the wait cap condition to be satisfied, $\bar{m}_i$ has to be the binding value or lower. (Even so, there is no Nash equilibrium of proposition 3 with lower $\bar{m}_i$. This is later proven by propositions 4.2 and 4.3.)

Proposition 4.

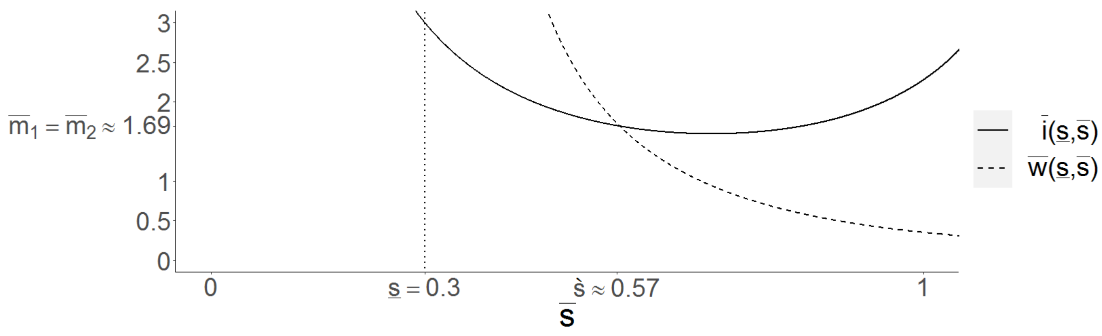

Figure 6: Nash Equilibria when $\underline{s} = 0.3$

1. *For any $\underline{s} > 0$, there exists a $\hat{s}$ that satisfies $\bar{i}(\underline{s},\hat{s}) = \bar{w}(\underline{s},\hat{s})$ which is unique in $(\underline{s},1)$. If $\bar{s} \in (\underline{s},\hat{s})$, $\bar{i}(\underline{s},\bar{s}) < \bar{w}(\underline{s},\bar{s})$. If $\bar{s} \in (\hat{s},1]$, $\bar{i}(\underline{s},\bar{s}) > \bar{w}(\underline{s},\bar{s})$.*

2. *Under assumption 3 for both players, assumption 4 for both players is a Nash equilibrium if and only if*

$$\bar{i}(\underline{s},\hat{s}) = \bar{w}(\underline{s},\hat{s}) \leq \bar{m}_1 = \bar{m}_2 = \bar{i}(\underline{s},\bar{s})$$

where $0 < \underline{s} < \bar{s} \leq \hat{s} < 1$.

3. *In the set of these Nash equilibria, $\bar{i}(\underline{s},\bar{s})$ is decreasing in $\bar{s}$.*

4. $\hat{s} - \underline{s} < \frac{1}{3}$.

I will now explain proposition 4. Propositions 3 and 4 describe the same set of symmetric pure strategy Nash equilibria. Specifically, for a given $\underline{s} > 0$, proposition 4 explains what Nash equilibria of proposition 3 can exist. For this explanation, $\hat{s}$, the intersection between the indifference condition curve, $\bar{i}(\underline{s},\bar{s})$ and the wait cap condition curve, $\bar{w}(\underline{s},\bar{s})$ is the key variable. I explain using figure 6 which depict $\bar{i}(\underline{s},\bar{s})$ and $\bar{w}(\underline{s},\bar{s})$ when $\underline{s} = 0.3$. Propositions 4.1$\sim$4.4 are verified by the figure.

Proposition 4.1 establishes that in $\bar{s} \in (\underline{s},1]$, $\bar{i}(\underline{s},\bar{s})$ and $\bar{w}(\underline{s},\bar{s})$ intersects exactly once. This intersection is $\hat{s}$. If $\bar{s}$ is smaller, $\bar{i}(\underline{s},\bar{s}) < \bar{w}(\underline{s},\bar{s})$ and if $\bar{s}$ is greater, $\bar{i}(\underline{s},\bar{s}) > \bar{w}(\underline{s},\bar{s})$. Proposition 4.2 states that there exists a Nash equilibrium of 3 where $\bar{m}_1 = \bar{m}_2 = \bar{i}(\underline{s},\bar{s})$ if $\bar{s}$ less than or equal to $\hat{s}$ but not if it is greater. Therefore propositions 4.1 and 4.2 are similar to proposition 3.

As proposition 3 states, in the figure, Nash equilibria of the proposition only exist on the $\bar{i}(\underline{s},\bar{s})$ curve where the indifference condition is fulfilled. Also, $\bar{m}_i$ must be below or on the $\bar{w}(\underline{s},\bar{s})$ curve so that the wait cap condition is fulfilled. In the figure, any point on the indifference condition curve where $\bar{s} \in (\bar{s},\hat{s}]$, supports a Nash equilibrium of the proposition whose $\bar{s}$ equals the x coordinate and whose $\bar{m}_1$ and $\bar{m}_2$ equal the y coordinate. Only these points supports a Nash equilibrium of the proposition. As proposition 4.3 states, in the set of these Nash equilibria, $\bar{i}(\underline{s},\bar{s})$ is decreasing in $\bar{s}$. Proposition 4.4 states that the distance between $\underline{s}$ and $\hat{s}$ is small. In other words, in the Nash equilibria, the distance between $\underline{s}$ and $\bar{s}$ is small.

In these Nash equilibria, for a fixed $\underline{s} > 0$, by the proposition 4.3, the higher $\bar{m}_1$ and $\bar{m}_2$ are, the lower $\bar{s}$ is. This means that the more players value the meeting, the lower the meeting probability, $(\bar{s} - \frac{(\bar{s}-\underline{s})^2}{2})^2$. Readers may find it odd that the more players value the meeting, the less they are willing to come in these Nash equilibria. This is explained by the wait cap condition. It is true that higher and higher values of $\bar{m}_1$ and $\bar{m}_2$ will eventually lead to a a higher $\bar{s}$. However, in all of the three figures, points on the indifference condition curve where

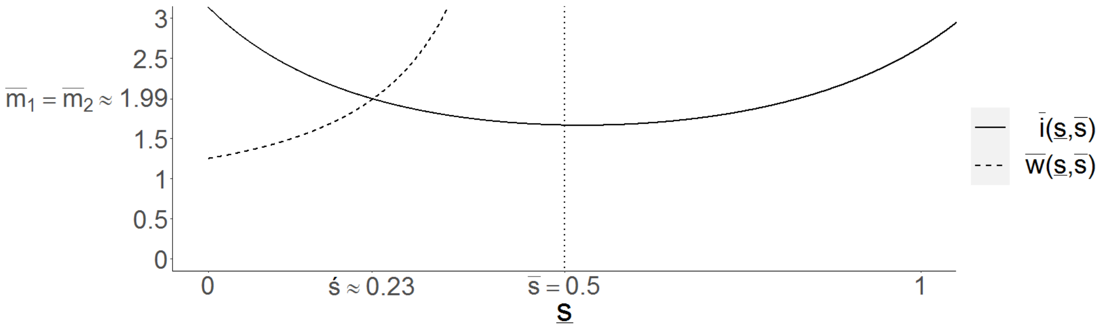

Figure 7: Nash Equilibria when $\bar{s} = 0.5$

$\bar{s}$ is near 1 or points above the indifference condition curve where $\bar{s} = 1$ do not support a Nash equilibrium as they are above the wait cap condition curve. (For the general case, this is proven by proposition 3 and lemma 26 in appendix 3.)

Proposition 5.

1. *For any $\bar{s} < 1$, there exists a $\acute{s}$ that satisfies $\bar{i}(\acute{s},\bar{s}) = \bar{w}(\acute{s},\bar{s})$ which is unique in $(0,\bar{s})$. If $\underline{s} \in [0,\acute{s})$, $\bar{i}(\underline{s},\bar{s}) > \bar{w}(\underline{s},\bar{s})$. If $\underline{s} \in (\acute{s},\bar{s})$, $\bar{i}(\underline{s},\bar{s}) < \bar{w}(\underline{s},\bar{s})$.*

2. *Under assumption 3 for both players, assumption 4 for both players is a Nash equilibrium if and only if*

$$\bar{m}_1 = \bar{m}_2 = \bar{i}(\underline{s},\bar{s}) \leq \bar{i}(\acute{s},\bar{s}) = \bar{w}(\acute{s},\bar{s})$$

where $0 < \acute{s} \leq \underline{s} < \bar{s} < 1$.

3. *In the set of these Nash equilibria, $\bar{i}(\underline{s},\bar{s})$ is decreasing in $\underline{s}$.*

4. *$\acute{s} - \underline{s} < \frac{1}{3}$.*

Propositions 3$\sim$ 5 all describe equivalent necessary and sufficient conditions for Nash equilibria satisfying assumption 4 for both players under assumption 3 for both players. Therefore, the Nash equilibria of proposition 5 also need to satisfy both the indifference condition and the wait cap condition. Note that propositions 4.1$\sim$4.4 respectively correspond to propositions 5.1$\sim$5.4.

I will explain proposition 5 using figure 7. For a fixed $\bar{s} < 1$, this figure depicts the indifference condition curve, $\bar{i}(\underline{s},\bar{s})$ and the wait cap condition curve, $\bar{w}(\underline{s},\bar{s})$ as $\underline{s}$ changes. The Nash equilibria of the proposition are on the indifference condition curve and cannot be above the wait cap condition curve. This means that the indifference condition curve only has Nash equilibria on $\underline{s} \in [\acute{s},\bar{s})$. ($\acute{s}$ is where the two curves intersect and $\underline{s} < \bar{s}$ by definition 2.) In these Nash equilibria, player's values of meeting, $\bar{m}_1$ and $\bar{m}_2$ are equal to the y coordinate of indifference condition curve. Since the Nash equilibria only exist on $\underline{s} \in [\acute{s},\bar{s})$, the distance between the earliest departure time, $\underline{s}$ and the latest departure time, $\bar{s}$ is less than $\frac{1}{3}$ in any of the Nash equilibria. Furthermore, in the set of the Nash equilibria, the indifference condition curve is downward sloping. This means that in the set of the Nash equilibria, the higher players' values of meeting are the lower the earliest departure time, $\underline{s}$.

Since, the meeting probability in the Nash equilibria is $(\bar{s} - \frac{(\bar{s}-\underline{s})^2}{2})^2$, for a fixed $\bar{s} < 1$, in the set of the Nash equilibria, the higher players' values of meeting are, the lower meeting probability. This is similar to the result I had in proposition 4. I will explain why higher values of meeting engender low meeting chance in the Nash equilibria. Recall that the player's strategies are symmetrical in the Nash equilibria. I fix $\bar{s} < 1$ and start from a Nash equilibria with high $\underline{s}$. Here, for low values of meeting, players are willing to have $\bar{s}$ as their latest departure time. Now, refer to figure 4. For this $\underline{s}$ and $\bar{s}$, if players' values of meeting, $\bar{m}_1$ and $\bar{m}_2$ are higher, this is no longer a Nash equilibrium because $E(m_i|d_i)$, player i's expected benefit for specific departure times would increase at $d_i = \bar{s}$. Therefore, players prefer to deviate to a higher latest departure time and increase their departure probabilities.

Next, suppose the players change their departure strategy so that they have a lower $\underline{s}$ but the same $\bar{s}$. This is a more demanding departure arrangement. Refer to figure 4 again. For the original players' values of meeting, $E(m_i|d_i)$ is below $E(c_i|d_i)$ at $d_i = \bar{s}$. Therefore, players prefer to deviate to a lower latest departure time. In other words, because they do not find it worthwhile to adhere to such a demanding departure arrangement and "fall off" by reducing arrival probability. In order for them to find it worthwhile to adhere to the departure arrangement, their values of meeting must increase when $\underline{s}$ decreases.

5 Discussion

5.1 Hazard rates and waits

The hazard rate of the other player's arrival often plays a key role in a player's wait decision. I will explain this informally. Usually, by comparing $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}\bar{m}_i$ to $\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}$, the player can figure out the sign of the marginal utility of actionable wait time, z_i. Here, $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}$ is the hazard rate of the other player's arrival which represents how likely the other player is to arrive marginally given that she has not arrived yet. To use this comparison, if the other player arrives first, she needs to wait until the player arrives. To restate, in deciding to wait marginally, the player looks at her values of meeting times the hazard rate of the other player's arrival as the marginal benefit of wait and compares it to her marginal cost of wait.

In propositions 1 and 3, travel times, r_i's are uniformly distributed and players' wait costs are linear. This leads to large planned wait times in the Nash equilibria of proposition 1 and the Nash equilibria of proposition 3. In these Nash equilibria, I can put a lower bound on the players' value of the meeting given that they come to the meeting place with positive probability and are willing to pay the travel cost. Also, upon arrival, players usually see an initially increasing hazard rate of the other player's arrival. In other words, usually, players initially calculate that the longer the player waits, the more likely the other player is to arrive marginally. Despite these two factors, the marginal cost of waiting is constant. These three factors combine to result in the players setting long planned abandonment times. Based on this logic, in general, when the hazard rates of players' arrival is increasing and the marginal costs of the players is increasing at a slower rate or not increasing, players are likely to plan to wait for a long time.

When the hazard rate of players' travel times is increasing, the hazard rate of players' arrival is increasing as well. The hazard rate is increasing for the normal distribution.[14] It is also increasing in the support of the PDF for the truncated normal distribution (lower tail truncated) and the uniform distribution.[15] I speculate that in most actual rendezvous, the marginal cost of waiting does not increase substantially until some time (at least 10 minutes) passes after the player arrives. Also, usually rendezvous have higher travel times than wait times which

14. See Nachlas (2017, p. 49–51)
15. See Nachlas (2017, p. 49-—51), Pham (2022, p. 98) and Oliveira et al. (2018, p. 174).

implies that people who value the meeting highly enough to travel to it are willing to wait for it. Therefore, in reality, people would be likely to plan to wait for a substantially long time.

5.2 Strategic complementarity of arrivals and planned waits

I will first explain the strategic complementary of arrivals. The model always has a trivial pure strategy Nash equilibrium where both player never come to the meeting place. Here, no player ever comes because the other player never comes. When the players' values of meeting are sufficiently high, this Nash equilibrium coexists with Nash equilibria where players come and meet with positive probability such as those in propositions 1 and 3.

Now I will discuss the set of Nash equilibria of proposition 3 using figure 6. Here, as shown in proposition 4, $\bar{s}$ is decreasing in the players' values of meeting, $\bar{m}_1$ and $\bar{m}_2$. This means that players' departure probability and meeting probability are decreasing in the players' values of meeting. In these Nash equilibria, $\bar{s} < 1$ is true and strategic complementary of arrival works to lesson the departure probability of both players. In other words, because a player does not always come in the Nash equilibria, the other player also chooses to not always come.

To see this in the figure, pick a point on the $\bar{i}(\underline{s},\bar{s})$ line where $\bar{s} \in (\underline{s},\hat{s}]$. (Recall that the $\bar{i}(\underline{s},\bar{s})$ line is where players are indifferent between departing for the meeting place at $\bar{s}$ and not departing) On this point, fix the values of the players values of meeting, $\bar{m}_1$ and $\bar{m}_2$ as $\bar{m}_1 = \bar{m}_2 = \bar{i}(\underline{s},\bar{s})$. Here, there exists a Nash equilibria of proposition 3. For a higher $\bar{s}$, $\bar{m}_1$ and $\bar{m}_2$ are above the $\bar{i}(\underline{s},\bar{s})$ line. This means that if any player deviates from the Nash equilibrium strategy to play a strategy where they depart even if they start later than $\bar{s}$, the other player will be willing to also depart even if they start later than $\bar{s}$. This demonstrates that both players are stuck at the Nash equilibria with low $\bar{s}$, arrival probability and meeting probability because the other player plays the strategy with the low $\bar{s}$.

The strategic complementary of planned waits is demonstrated in the Nash equilibria of proposition 1. Here, the meeting probability is 1 and players wait till the other player arrives with probability 1. This behavior is because of how the strategic complementary of planned waits works in conjunction with the hazard rate of the uniform distribution and linear wait cost. (The earlier subsection explains the effect of the hazard rate of the uniform distribution and linear wait cost.) In general, the hazard rate of the uniform distribution and linear wait cost explain why for specific arrival times, players set large planned wait times that players wait till the other player comes with probability 1. On the other hand, the strategic complementary of planned waits generally explains why players do set such large planned wait times for all material arrival times.

I will describe the logic loosely. Given that the players have a positive probability of arrival in the Nash equilibrium, I can put a lower bound on the players' values of meeting. Given this lower bound, I reason that in the Nash equilibrium, there is some minimum amount of planned wait. The following explains the strategic complementary of planned waits. When a player plans to wait for the other player, this provides an incentive for the other player to wait as well. Now, since the player plans to wait for the other player, if the other player decides to wait, it will not be futile. Rather, if the player has not come yet, she will do so in the future. Conversely, if the player plans to not wait, the other player's wait might be futile as the player may have already left earlier. Therefore, the initial minimum amount of planned wait leads to successively more and more planned wait by players. This leads to the result that players wait till the other player arrives with probability 1.

5.3 Meeting values and departure times

In the Nash equilibria of proposition 1, the player who departs earlier is not necessarily the player who values the meeting more. Once the lower bound conditions of the proposition's (1) and (2) are met, the players' values for the meeting can be arbitrarily higher. Proposition 2 reveals that in these Nash equilibria, the comparatively earlier the player departs, the higher her expected cost. Therefore, players want to depart late and have the other player wait for them. Proposition 2's (1) shows that the sum of the players' expected costs is increasing in the absolute value of the difference in players' departure times. The player who departs earlier incurs excessive expected wait cost from a social welfare perspective and the more the players departure times differ, the lower the social welfare.

In the Nash equilbria of proposition, a player who departs early is revealed to have a high value of the meeting. This is because she is willing to pay the high expected wait cost. On the contrary, the value of the meeting is not revealed for the player who departs late. Because of this, disclosing a high value of the meeting can adversely affect the discloser in the ensuing rendezvous game. Statements such as "You must come." can inform the listener that the speaker's value of the meeting is high. Thus, the listener might depart late and have the speaker wait for her.

This disadvantage of a disclosed high value of the meeting also applies to meetings involving the head of states. Vladimir Putin has been known to be habitually late to meetings between heads of states.[16] Despite this, there is no known case of a head of a state giving up on a meeting with Putin and going elsewhere.[17] In 2020, Recep Tayyip Erdogan waited in the Kremlin to meet with Putin. In 2022, Putin was shown on camera awkwardly waiting for Erdogan for their meeting in Iran.[18] He was also seen in Uzbekistan waiting for leaders of Turkey, Azerbaijan, India and Kyrgyzstan.[19] When Putin met Donald Trump in Helsinki, after learning that Putin would be late, Trump who was already set to arrive late, delayed his own departure even further.[20] When Mao Zedong visited Joseph Stalin, Stalin made Mao wait for weeks outside Moscow.[21] Barack Obama has also been known for habitually being late to events.[22]

These accounts lend credence to the theory that by arriving late, these leaders are executing a deliberate strategy to make the other party wait for them. This explains why Trump delayed departure to the meeting after learning that Putin would be late. Because leaders of nations are aware that the other party values meeting with them highly and will wait for them, they can deliberately delay departure and make the other party wait. While the leaders themselves may also value the meeting highly, they may conceal this fact when they depart late.

5.4 Remedial compensations

In the Nash equilibria of proposition 3, $\bar{s} < 1$ holds which means that players do not always come to the meeting and that the meeting probability is low. Why is there no Nash equilibria of the proposition with $\bar{s} = 1$ and a higher meeting probability? I will explain using figure 6. There is no Nash equilibrium of the proposition when $\underline{s} = 0$. Suppose that when $\underline{s} > 0$ and $\bar{s} = 1$, $\bar{m}_1 = \bar{m}_2$ is on or above the $\bar{l}(\underline{s}, \bar{s})$ line. (A Nash equilibria of the proposition with $\bar{s} = 1$

16. Ma (2019), Jankowicz (2022), and Batchelor (2017)
17. Walker (2015)
18. Jankowicz (2022)
19. Landen (2022)
20. Herszenhorn and Karni (2018), Korte and Fritze (2018), and Meredith (2018)
21. Lau (2022)
22. Bump (2014)

has to be on or above the $\bar{l}(\underline{s},\bar{s})$ line so that the players are willing to depart at $\bar{s}$.) The figures show that this does not necessary require that $\bar{m}_1 = \bar{m}_2$ be higher than the level at the Nash equilibria of the proposition. When $\underline{s}$, $\bar{s}$, $\bar{m}_1$ and $\bar{m}_2$ take on the values, the wait cap condition, $\forall i \in \{1,2\}, \bar{m}_i \leq \bar{w}(\underline{s},\bar{s})$ is violated so players prefer to wait beyond $\underline{s}+1$ when they arrive.

Another way to see this is to look at figures 3. In figure 3, increasing $\bar{s}$ also increases the size of the C triangle and the probability players arrive after $\underline{s}+1$. Therefore, waiting beyond $\underline{s}+1$ becomes more attractive. In other words, if players always arrive at the meeting with $\bar{s}=1$, they also prefer some strategy which has planned wait beyond $\underline{s}+1$.

If a player is known to wait beyond $\underline{s}+1$ when she arrives, the other player has no incentive to depart as early as $\underline{s}$. The other player can delay her departure slightly from $\underline{s}$ knowing that if the player arrived earlier, the player will wait till she arrives. From the Nash equilibria, there are two ways of increasing meeting probability. The first is increasing arrival probability by setting $\bar{s}=1$. The second is increasing a player's wait by increasing the planned abandonment time, $\zeta_i = \underline{s}+1$ when the player arrives. Both run into the problem that the other player will respond by delaying departure from $\underline{s}$. This leads to players' strategies no longer being symmetric. A player wants the other player to come early. To provide an incentive for the other player to arrive early, she raises the danger of missed meeting by limiting her own planned abandonment time to never exceed $\zeta_i = \underline{s}+1$.

As a remedy for the problem of low meeting probability, I propose compensating players for arriving early and waiting. Suppose that players are compensated thusly, $\underline{s} > 0$ and $\bar{s} = 1$ hold and players always arrive. Now given the compensation, players may decide on $\zeta_i = \bar{s}+1 = 2$ instead of $\zeta_i = \underline{s}+1$. When a player always plays $\zeta_i = \bar{s}+1 = 2$ and always waits till the other player arrives, the other player still has an incentive to come early because if she arrives later, she has to pay compensation. This demonstrates how compensation can lead to the players to move to a Nash equilibrium with a meeting probability of 1. Since the meeting chance increases and symmetric transfers do not directly effect the sum of players' utility, both players can be better off.

In reality, monetary compensations for waits may be socially awkward and indecorous especially if the players are acquaintances. Likewise, if restaurants charge customers fees for missing reservations or being late the reservations, the customers may feel antipathy, rate the restaurant badly or go to a different restaurant. In cases where such monetary transfers are difficult to implement or onerous, players may consider non-monetary transfers instead. For instance, if friends are meeting for a meal, they could agree that the person who arrives late pays for the meal or that the person who arrives early picks the restaurant. Similarly, if the friends are seeing a movie together, the person who arrives first could pick the movie. In case of restaurant reservations, restaurants could waive tips or offer lagniappes such as free drinks for customers who come early.

5.5 Application to transfers in supply chains

While compensating players for arriving early and waiting may be Pareto superior, fines that go beyond compensations may decrease social welfare. I will explain this focusing on the application to supply chain settings. In supply chains, liquidated damages stated in a contract is the monetary compensation for estimated loss that the party that violates the contracts must pay. (For legal analyses of liquidated damages, refer to Brizzee (1991) and Goetz and Scott (1977)) In the United States, for a liquidated damages provision to be enforceable, it must not be a penalty, i.e. the compensation must be a reasonable estimate of the loss and the payment can not be a punishment for the breach of contract. I raise the argument that in supply chain settings, unilateral penalty provisions that go beyond estimated damages can result in a decrease in social welfare.

The Nash equilibria of proposition 1 applies to the supply chain setting in the following way. From the upstream's perspective, departure corresponds to the firm starting work on the project or the product contracted by the downstream. Arrival corresponds to the firm finishing the contracted work on this project or product. This can be delivery or installation of the product. Wait time corresponds to the time from the completion of the upstream's work to when the downstream firm actually makes use of what the upstream completed.

From the downstream's perspective, departure means the downstream begins preparations for making use of the upstream's project or product. This preparation can be making space in its shelves or warehouses to place the product. It can also be readying the environment for the upstream's work or the installation of the product. In other cases, the downstream might prepare parts or equipment it will use in conjunction with the upstream's product. Arrival means completion of the preparations. Wait time is the time from the completion of the preparations to when the downstream actually starts makes use of the product or project from the upstream firm.

The meeting succeeds when the downstream firm receives the upstream firm's project or product and starts to make use of it. For instance, if the downstream starts using the parts from the upstream firm in assembly, that corresponds to a successful meeting. If the downstream firm displays and starts selling the product it receives from the upstream firm, that also corresponds to a meeting.

Consider the following unilateral penalty. If player 2 arrives late, she pays player 1 but player 1 never pays player 2. Player 2 needs a high value of the meeting for her to depart comparatively early and pay the high expected cost. A high fine on player 2's late arrival, provides the incentive for player 2 to not delay departure. By having player 2 depart early and wait for player 1, player 1 extracts player 2's surplus. In a supply chain setting, player 2's value of the meeting would be mostly determined by the payment for the fulfillment of the contract. For player 1, unlike raising this payment to lower player 2's departure time, raising player 2's fine for late arrival is costless and also guarantees that player 2 cannot depart comparatively late.

Liquidated damages can compensate players for their wait costs. As discussed in the previous subsection, such compensations can make both player being better off. However, unlike liquidated damages, unilateral penalty provisions can engender asymmetric equilibria where a firm extracts the other firm's surplus and social welfare is reduced because of this. Therefore, prohibiting penalty provisions can increase social welfare by leading to Nash equilibria where the upstream firm and the downstream firm's expected wait times are more similar.

5.6 Coordination of meetings

A problem with real-life rendezvous is that when people decide on the meeting time, it is unclear exactly what this time is. Is it the earliest time or the latest time that players might meet in or something else? My assertion is that when players agree on the meeting time they often implicit set the latest times in which players might depart even if the promised time does not actually equal these times. My evidence is that often, after setting the meeting time, people decide how early to depart to not be late. In the Nash equilibria of proposition 3, this corresponds to starting with a fixed $\bar{s}$ and setting a earlier $\underline{s}$ when the players' values of meeting are high. When $\bar{s} < 1$ is fixed, proposition 5 and figure 7 show that this is how players act in these Nash equilibria. The problem is that meeting probability, $(\bar{s} - \frac{(\bar{s}-\underline{s})^2}{2})^2$ is increasing in $\underline{s}$. Therefore, when players initially fix $\bar{s}$, the more players value the meeting, the lower the meeting probability. Intuitively, when players value the meeting more and depart at earlier times, since both players expect the other player to arrive earlier, both will abandon the meeting place earlier than before.

Suppose players initially fix $\underline{s} > 0$ instead of $\bar{s}$. In this case, as proposition 4 and figure 6 show, in the set of these Nash equilibria, $\bar{s}$ is decreasing in the players' values of the meeting. Meeting probability, $(\bar{s} - \frac{(\bar{s}-\underline{s})^2}{2})^2$ is increasing in $\bar{s}$. Therefore, higher values of meeting lead to a lower meeting probability in this case also. However, $\bar{s}$ cannot be $\underline{s}$ or smaller. This guarantees that when players initially fix $\underline{s}$, the infimum of meeting probability is $\underline{s}^2$.

One way people can move to a Nash equilibrium with high meeting probability is the following script. A person may start by asking the question of "When is a good time for you to meet?". After the two people find a meeting time at which they can arrive with high reliability, they could promise that "We won't come early but we will wait moderately".

When players are constrained by start time variation, this script can lead them to a Nash equilibrium of proposition 3 and a meeting probability greater than $\underline{s}^2$. By saying, "We won't come early", players avoid fixing $\bar{s}$ first and adjusting $\underline{s}$ downwards. Instead, players decide when it becomes too late for them to depart. This corresponds to setting $\bar{s}$ when $\underline{s}$ is fixed. In cases where players can avoid being constrained by start time variation, this script may lead them to a symmetric Nash equilibrium of proposition 1 where players always meet. This is because if they can always depart at a certain time, they can play a pure strategy with a fixed departure time. Furthermore, players can avoid asymmetric Nash equilibria with reduced social welfare because they both agree to wait moderately.

6 Conclusion

In this paper, I solve for the pure strategy Nash equilibria of the continuous rendezvous game. For doing this, I find a player's optimal wait time decision by comparing the other player's hazard rate of arrival multiplied by the player's value of the meeting to the player's marginal cost of wait. The Nash equilibria of proposition 1 are for when there is no start time variation. In these Nash equilibria, players always wait till the other player arrives and always meet. Also, as long as the players' values of the meeting are sufficiently high, any player can be the first to depart even if she values the meeting less. In meetings involving the heads of nations, the heads may use this fact to intentionally delay departure knowing that others who value the meeting will wait. The Nash equilibria of proposition 3 are for the case where start time variation exists. These Nash equilibria are characterized by $\underline{s}$, the earliest departure time and $\bar{s}$, the latest departure time. In these Nash equilibria, the meeting probability is always less than 1. This is because the possibility of a missed meeting is required to induce the players to depart early.

To deal with this issue, I propose compensating the person who arrives early and waits. When monetary compensations violate social norms, non-monetary compensations such letting the person where to dine may work. While such compensations can move people to Pareto superior equilibria, unilateral punishments that go beyond compensation may lower social welfare.

If players who value the meeting highly first agree on $\bar{s}$ and then set a small $\underline{s}$, they have a low meeting probability in the Nash equilibria of proposition 3. For high meeting probability, players should agree to not come early but to wait for each other. In the Nash equilibria, this corresponds to initially agreeing on $\underline{s}$ and then setting $\bar{s}$.

Appendix 1. Intermediate results and proofs

The lemmas and propositions that are stated and/or proven here are about the basic attributes of the game and are used elsewhere to derive other results. When any of the four equivalent conditions in the lemma below is satisfied, the players meet. Reformulating the meeting condition helps prove many other results.

Lemma 2.

$$max\{a_1,a_2\} \leq min\{z_1,z_2\} \tag{7}$$

$$\leftrightarrow$$

$$a_1 = a_2,\ a_1 \leq a_2 \leq \zeta_1 \ or\ a_2 \leq a_1 \leq \zeta_2 \tag{8}$$

$$\leftrightarrow$$

$$a_1 \leq a_2 \leq z_1 \ or\ a_2 \leq a_1 \leq z_2 \tag{9}$$

$$\leftrightarrow$$

$$a_2 \leq z_1 \ and\ a_1 \leq z_2 \tag{10}$$

Proof. I will first prove (7) $\rightarrow$ (8). Suppose $a_1 = a_2$. The consequent holds. Using symmetry, suppose $a_1 < a_2$. $a_2 \leq z_1 \leftrightarrow a_2 \leq max\{a_1,\zeta_1\} \rightarrow a_2 \leq \zeta_1$. Thus $a_1 \leq a_2 \leq \zeta_1$.

Now I will prove (8) $\rightarrow$ (9). If $a_1 = a_2$, $a_1 = a_2 \leq z_1$. Using symmetry, if $a_1 \leq a_2 \leq \zeta_1$, $a_1 \leq a_2 \leq \zeta_1 \leq z_1$.

Now I will prove (9) $\rightarrow$ (7). Using symmetry, if $a_1 \leq a_2 \leq z_1$, $a_1 \leq a_2 \leq z_2$.

Equations 7, 8 and 9 are equivalent.

Now I will prove (7) $\rightarrow$ (10). If equation 7 holds, $max\{a_1,a_2\} \leq z_1$ and $max\{a_1,a_2\} \leq z_2$.

Now I will prove (10) $\rightarrow$ (9). Using symmetry, if equation 10 holds and $a_1 \leq a_2$, $a_1 \leq a_2 \leq z_1$. ∎

The following proposition is used in marginal analysis. The proposition's (1) is used to state the marginal expected utility of actionable abandonment time, z_i. The proposition's (2) is used to find the sign of the marginal expected utility of actionable abandonment time, z_i.

Proposition 6. *For any values of d_i, a_i and z_i, let δ be a proper interval containing z_i satisfying $a_i < \sup\delta$. Suppose γ_{-i} and $\frac{\partial c_i}{\partial w_i}$ exist. If γ_{-i} is continuous when the domain is δ and in partially differentiating $E(u_i|d_i,a_i,z_i)$, the domain of the variable of interest is set to δ, the following holds.*

(1)

$$\frac{\partial E(u_i|d_i,a_i,z_i)}{\partial z_i} = \gamma_{-i}(z_i)\bar{m}_i - (E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i) + 1 - \Gamma_{-i}(z_i))\frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i}.$$

(2) If $E(M|a_i,z_i) < 1$,

$$\frac{\partial E(u_i|d_i,a_i,z_i)}{\partial z_i} \lesseqqgtr 0$$

$$\leftrightarrow$$

$$\frac{\gamma_{-i}(z_i)}{E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i) + 1 - \Gamma_{-i}(z_i)}\bar{m}_i - \frac{\partial c_i(a_i-d_i,z_i-a_i)}{\partial z_i} \lesseqqgtr 0. \tag{11}$$

Proof. Using symmetry, I say $i = 1$. Suppose c_1 is differentiable in w_1, γ_2 exists and γ_2 is continuous in δ. Fix d_1, a_1, z_1 and $z_1' \geq z_1$ so that they are possible values and $\{z_1, z_1'\} \subset \delta$. The following definition slightly abuses notation.

$$\triangle(z_1', z_1) \equiv E(u_1|d_1, a_1, z_1', a_2) - E(u_1|d_1, a_1, z_1, a_2)$$

If $a_2 \leq a_1 \leq z_2$, $\triangle(z_1', z_1) = 0$.
If $a_2 \leq z_2 < a_1$, $\triangle(z_1', z_1) = -c_1(a_1 - d_1, z_1' - a_1) + c_1(a_1 - d_1, z_1 - a_1)$.
If $a_1 \leq a_2 \leq z_1$, $\triangle(z_1', z_1) = 0$.
If $a_1 \leq z_1 < a_2 \leq z_1'$, $\triangle(z_1', z_1) = \bar{m}_1 - c_1(a_1 - d_1, a_2 - a_1) + c_1(a_1 - d_1, z_1 - a_1)$.
If $a_1 \leq z_1 \leq z_1' < a_2$, $\triangle(z_1', z_1) = -c_1(a_1 - d_1, z_1' - a_1) + c_1(a_1 - d_1, z_1 - a_1)$.

If γ_2 exists, $P(a_1 = a_2) = 0$.

$$\begin{aligned}
&E(u_1|d_1, a_1, z_1') - E(u_1|d_1, a_1, z_1) = \bar{m}_1 P(a_1 \leq z_1 < a_2 \leq z_1') \\
&- (P(a_2 \leq z_2 < a_1) + P(a_1 \leq z_1 \leq z_1' < a_2))(c_1(a_1 - d_1, z_1' - a_1) - c_1(a_1 - d_1, z_1 - a_1)) \\
&- \int_{z_1}^{z_1'} (c_1(a_1 - d_1, x - a_1) - c_1(a_1 - d_1, z_1 - a_1))\gamma_2(x)\,dx \\
&= \bar{m}_1 \int_{z_1}^{z_1'} \gamma_2(x)\,dx \\
&- (P(z_2 < a_1) + 1 - \Gamma_2(z_1'))(c_1(a_1 - d_1, z_1' - a_1) - c_1(a_1 - d_1, z_1 - a_1)) \\
&- \int_{z_1}^{z_1'} (c_1(a_1 - d_1, x - a_1) - c_1(a_1 - d_1, z_1 - a_1))\gamma_2(x)\,dx
\end{aligned}$$

By the fundamental theorem of calculus, $\Gamma_{-i}'(z_i') = \gamma_{-i}(z_i')$ when the domain of z_i' is δ for the differentiation. Therefore, by the Leibniz integral rule, for the same domain for differentiation,

$$\frac{\partial E(u_i|d_1, a_1, z_1')}{\partial z_1'} = \bar{m}_1 \gamma_2(z_1') - (P(z_2 < a_1) + 1 - \Gamma_2(z_1')) \frac{\partial c_1(a_1 - d_1, z_1' - a_1)}{\partial z_1'}.$$

This proves (1). Now I will prove (2) with the fixed values from (1).

$$P(z_2 < a_1) + 1 - \Gamma_2(z_1) = P(z_2 < a_1) + P(z_1 < a_2) \tag{12}$$

Suppose $z_2 < a_1$ and $z_1 < a_2$. If $a_1 \leq a_2$, $a_1 \leq z_2$. If $a_2 < a_1$, $a_2 \leq z_1$. Thus $\{z_2 < a_1\}$ and $\{z_1 < a_2\}$ are disjoint sets.

$$\begin{aligned}
&P(z_2 < a_1) + P(z_1 < a_2) = \\
&\qquad P(\{z_2 < a_1\} \cup \{z_1 < a_2\}) = 1 - P(\{a_1 \leq z_2\} \cap \{a_2 \leq z_1\}) = 1 - E(M)
\end{aligned} \tag{13}$$

Here, the last equality is by lemma 2 and equation 10. By equations 12 and 13, I have the following.

$$P(z_2 < a_1) + 1 - \Gamma_2(z_1) = 1 - E(M|a_1, z_1) \tag{14}$$

■

Lemma 3. *Suppose g_1 and g_2 exist.*

$$E(m_i|d_i,d_{-i},\zeta_i,\zeta_{-i}) = \bar{m}_i(\int_{d_i}^{\zeta_i} G_i(x-d_i)g_{-i}(x-d_{-i})\,dx \\ + \int_{d_{-i}}^{\zeta_{-i}} g_i(x-d_i)G_{-i}(x-d_{-i})\,dx) \tag{15}$$

Proof. Using symmetry, I will prove for $i=1$. $E(m_1|d_1,d_2,\zeta_1,\zeta_2)$ exists by Lebesgue's dominated convergence theorem. Consider $E(m_1)$ in a game where d_1, d_2, ζ_1 and ζ_2 are fixed. For such a game, we have the following.

$$E(m_1) = \int_{a_1<a_2} m_1\,dP + \int_{a_1>a_2} m_1\,dP$$

Consider $a_1 < a_2$. $P(z_1 = a_2) = 0$. $z_1 < a_2$ means player 2 arrives after player 1's actionable abandonment time. Therefore, if $z_1 < a_2$, $m_1 = 0$.

$$\begin{aligned}\int_{a_1<a_2} m_1\,dP &= \int_{\{a_1<a_2\}\cap\{a_2<z_1\}} m_1\,dP = \int_{a_1<a_2<z_1} m_1\,dP \\ &= \int_{\{a_1<a_2<z_1\}\cap\{a_1<\zeta_1\}} m_1\,dP + \int_{\{a_1<a_2<z_1\}\cap\{a_1\geq\zeta_1\}} m_1\,dP\end{aligned}$$

$a_1 \geq \zeta_1$ means $z_1 = a_1$. $P(a_1 < a_2 \leq a_1)$=0. $a_1 < \zeta_1$ means $z_1 = \zeta_1$.

$$\begin{aligned}\int_{a_1<a_2} m_1\,dP &= \int_{\{a_1<a_2<z_1\}\cap\{a_1<\zeta_1\}} m_1\,dP = \bar{m}_1 P(a_1 < a_2 < \zeta_1) \\ &= \int_{d_1}^{\zeta_1}\int_{x}^{\zeta_1} m_1 g_1(x-d_1)g_2(y-d_2)\,dy\,dx \\ &= m_1 \int_{d_1}^{\zeta_1} g_1(x-d_1)\int_{x}^{\zeta_1} g_2(y-d_2)\,dy\,dx\end{aligned}$$

$$\int_{x}^{\zeta_1} g_2(y-d_2)\,dy = G_2(\zeta_1-d_2) - G_2(x-d_2)$$

CDFs are monotonic and therefore Riemann-integrable. PDFs are also Riemann-integrable. Thus, products of a CDF and a PDF are Riemann-integrable.[23]

$$\begin{aligned}\int_{d_1}^{\zeta_1} g_1(x-d_1)\int_{x}^{\zeta_1} g_2(y-d_2)\,dy\,dx &= \int_{d_1}^{\zeta_1} g_1(x-d_1)(G_2(\zeta_1-d_2) - G_2(x-d_2))\,dx \\ &= G_2(\zeta_1-d_2)\int_{d_1}^{\zeta_1} g_1(x-d_1)\,dx \\ &\quad - \int_{d_1}^{\zeta_1} g_1(x-d_1)G_2(x-d_2)\,dx \\ &= G_1(\zeta_1-d_1)G_2(\zeta_1-d_2) \\ &\quad - \int_{d_1}^{\zeta_1} g_1(x-d_1)G_2(x-d_2)\,dx \\ &= \int_{d_1}^{\zeta_1} G_1(x-d_1)g_2(x-d_2)\,dx\end{aligned}$$

23. See Apostol (1985, p. 128,158-159).

Since G_1 and G_2 have PDF's, they are absolutely continuous. Therefore, I can use integration by parts for the last equality.[24]

I now have $\int_{a_1<a_2} m_1\,dP = \int_{d_1}^{\zeta_1} G_1(x-d_1)g_2(x-d_2)\,dx$. Recall that $m_1 = \bar{m}_1$ if and only if the meeting succeeds. Therefore, symmetry gives $\int_{a_1>a_2} m_1\,dP = \int_{d_2}^{\zeta_2} G_2(x-d_2)g_1(x-d_1)\,dx$.

$$E(m_1) = \bar{m}_1\left(\int_{d_1}^{\zeta_1} G_1(x-d_1)g_2(x-d_2)\,dx + \int_{d_2}^{\zeta_2} g_1(x-d_1)G_2(x-d_2)\,dx\right) \tag{16}$$

■

The above lemma calculates the expected benefit of the game using integration. To understand lemma 3, I can refer to lemma 2 and formula 8. In a continuous setting like this one where the probability of the players meeting by arriving at exactly the same time is 0, I only need consider two scenarios of a successful meeting. $a_i \leq a_{-i} \leq \zeta_i$ includes the scenario where player i comes and player $-i$ comes after player i but before player i abandons the meeting place. $a_{-i} \leq a_i \leq \zeta_{-i}$ includes the scenario where player $-i$ comes and player i comes after player $-i$ but before player $-i$ abandons the meeting place. I assume $a_i \neq a_{-i}$ for now and this means that the $a_i \leq a_{-i} \leq \zeta_i$ and $a_{-i} \leq a_i \leq \zeta_{-i}$ are mutually exclusive. One player must come first and $a_i < a_{-i}$ and $a_{-i} < a_i$ cannot be true at the same time. In order for $a_i \leq a_{-i} \leq \zeta_i$ to happen, player $-i$ needs to arrive between d_i and ζ_i (inclusive). Also, player i needs to arrive before player $-i$. This explains the $\int_{d_i}^{\zeta_i} G_i(x-d_i)g_{-i}(x-d_{-i})\,dx$ part of equation 15. The $\int_{d_{-i}}^{\zeta_{-i}} g_i(x-d_i)G_{-i}(x-d_{-i})\,dx$ part is explained in a similar way.

Appendix 2. Lemma and proofs used in subsection 4.1

Lemmas used in this section but not found in this paper are in chapter 1 of the Supplemental Material.

Proposition 7. *Under assumptions 1 and 2, the following for some i is necessary and sufficient for a pure strategy Nash equilibrium with $E(M) > 0$.*

(1) $\bar{m}_i \geq max\{\frac{(d_i-d_{-i})^2+1}{2(d_i+1-d_{-i})}, \frac{1}{2} + \frac{(d_i+1-d_{-i})^3}{6} + d_{-i} - d_i\}$

(2) $\bar{m}_{-i} \geq \frac{1}{2} + \frac{(d_i+1-d_{-i})^3}{6}$

(3) $d_i \leq d_{-i} < d_i + 1$.

(4) $P(\zeta_i = d_{-i} + 1) = 1$

(5) $P(\{\zeta_{-i} \neq d_i + 1\} \cap \{a_{-i} \in [d_{-i}, d_i + 1)\}) = 0$

(6) *If* $a_{-i} \geq d_i + 1$, $z_{-i} = a_{-i}$.

Proof. By lemma 1, $\bar{m}_i \geq 0.5$ and $\bar{m}_{-i} \geq 0.5$ are required. Using symmetry, I assume $d_i \leq d_{-i}$. I start from the properties of lemma 16. I will look into z_i^*. By lemma 10, if $d_i < d_{-i}$, unless the following equation holds, lemma 16's (2) is not optimal. The $d_i = d_{-i}$ case is trivial.

$$d_i \geq d_{-i} - \bar{m}_i + 0.5 \tag{17}$$

24. See Cohn (2013, p. 135–137,173–174).

If formula 17 holds, I have the following.

$$\forall a_i \in [d_i, d_{-i}] : d_{-i}+1 \in z_i^*(a_i) \tag{18}$$

When $a_i \in [d_{-i}, d_i+1]$, lemma 16's (3) means following.

$$P(z_{-i} < a_i) = 0.$$

$$P(z_{-i} < a_i) + \frac{d_{-i}+1-a_i}{2} \leq \frac{1}{2} \leq \bar{m}_i$$

Therefore, the following holds by lemma 11.

$$\forall a_i \in [d_{-i}, d_i+1] : d_{-i}+1 \in z_i^*(a_i) \tag{19}$$

By lemma 12, I have the following.

$$\forall a_i \in [d_{-i}, min\{2\bar{m}_i + d_{-i} - 1, d_{-i}+1\}) : d_{-i}+1 \in z_i^*(a_i) \tag{20}$$

Now, I will look into z_{-i}^*. When $a_{-i} \in [d_{-i}, d_i+1]$, lemma 16's (2) means following.

$$P(z_i < a_{-i}) = 0.$$

$$P(z_i < a_{-i}) + \frac{d_i+1-a_{-i}}{2} \leq \frac{1}{2} \leq \bar{m}_{-i}$$

Therefore, the following holds by lemma 11.

$$\forall a_{-i} \in [d_{-i}, d_i+1] : d_i+1 \in z_{-i}^*(a_{-i}) \tag{21}$$

If $a_{-i} \in [d_i+1, d_{-i}+1]$, $P(a_i \leq a_{-i} \leq z_i) = 1$ and by lemma 19, I have the following.

$$\forall a_{-i} \in [d_i+1, d_{-i}+1] : a_{-i} \in z_{-i}^*(a_{-i}) \tag{22}$$

$a_{-i} > d_{-i}+1$ is impossible.

By what I have figured out till now about z_i^* and z_{-i}^*. I know that when formula 17 is satisfied, the players find z_i and z_{-i} of the Nash equilibrium optimal.

Next, I will look at d_i. I will find player i's utility in the Nash equilibrium. In the Nash equilibrium, by lemma 2 and formula 9,

$$E(M) = 1. \tag{23}$$

The following is player i's cost in the Nash equilibrium.

$$\begin{aligned}
&E(c_i) = E(r_i) + E(w_i) = \\
&\frac{1}{2} + \int_{a_i \leq d_{-i}} a_{-i} - a_i \, dP + \int_{d_{-i} \leq a_i \leq d_i+1, d_{-i} \leq a_{-i} \leq d_i+1} max\{0, a_{-i} - a_i\} \, dP \\
&+ \int_{d_{-i} \leq a_i \leq d_i+1, d_i+1 \leq a_{-i}} a_{-i} - a_i \, dP = \\
&\frac{1}{2} + \int_{d_i}^{d_{-i}} \int_{d_{-i}}^{d_{-i}+1} y - x \, dy \, dx + \int_{d_{-i}}^{d_i+1} \int_{x}^{d_i+1} y - x \, dy \, dx \\
&+ \int_{d_{-i}}^{d_i+1} \int_{d_i+1}^{d_{-i}+1} y - x \, dy \, dx = \\
&\frac{1}{2} + (d_{-i}+1-d_i)\frac{d_{-i}-d_i}{2} + \frac{(d_i+1-d_{-i})^3}{6} + (d_i+1-d_{-i})\frac{d_{-i}-d_i}{2} = \\
&\frac{1}{2} + \frac{(d_i+1-d_{-i})^3}{6} + d_{-i} - d_i
\end{aligned} \tag{24}$$

Therefore, in order for player i to weakly prefer coming to the meeting, the following condition is required.

$$\bar{m}_i \geq \frac{1}{2} + \frac{(d_i+1-d_{-i})^3}{6} + d_{-i} - d_i$$

Note that this condition makes the condition imposed by formula 17 unnecessary.

Fix the value of d_i in the Nash equilibrium as d. By lemma 7, $d_i < d$ is not optimal. By lemma 17's (1), $d_i > d_{-i}$ is not optimal. In finding the optimal d_i, I only need consider $d_i \in [d, d_{-i}]$. Consider the case where $d_{-i}+1 \leq 2\bar{m}_i + d_{-i} - 1$. By formula 20, I have the following.

$$\forall a_i \in [d_{-i}, d_i+1] : d_{-i}+1 \in z_i^*(a_i) \tag{25}$$

Consider the case where $2\bar{m}_i + d_{-i} - 1 < d_{-i}+1$. This means $\bar{m} < 1$. In this case, by lemma 17's (2), $d_i > max\{d, 2\bar{m}_i + d_{-i} - 2\}$ is not optimal for player i. If $d \geq 2\bar{m}_i + d_{-i} - 2$, $d_i > d$ is not optimal. If $d < 2\bar{m}_i + d_{-i} - 2$, $d_i > 2\bar{m}_i + d_{-i} - 2$ is not optimal. If $d_i \leq 2\bar{m}_i + d_{-i} - 2$, $d_i + 1 \leq 2\bar{m}_i + d_{-i} - 1$. So by formula 20, formula 25 holds. Therefore, by formulas 18 and 19, I only need check whether any of the departure times $d_i \in [d, d_{-i}]$ with z_i fixed as $d_{-i}+1$ is better than $d_i = d$ for player i. Consequently, I will find the probability of meeting and the cost of player i under the conditions that $d \leq d_i \leq d_{-i}$, $z_i = d_{-i}+1$ and $z_{-i} = d+1$.

$$E(M) = \int_{d_{-i}}^{d_i+1} x - d_i \, dx + \int_{d_i+1}^{d_{-i}+1} dx + \int_{d_{-i}}^{z_{-i}} x - d_{-i} \, dx = \\ \frac{1}{2} - \frac{(d_{-i}-d_i)^2}{2} + d_{-i} - d_i + \frac{(z_{-i}-d_{-i})^2}{2}$$

Here, the first equality is by lemma 3. I also find derivatives for the case where, in addition to the conditions above, $d < d_{-i}$ also holds.

$$\frac{\partial E(M)}{\partial d_i} = d_{-i} - d_i - 1 \tag{26}$$

$$\frac{\partial^2 E(M)}{\partial d_i^2} = -1 \tag{27}$$

$$\begin{aligned} E(c_i) = &\int_{a_{-i} \leq a_i, z_{-i} \leq a_i} z_i - d_i \, dP + \int_{a_i \leq a_{-i}, a_{-i} \leq d_i+1} a_{-i} - d_i \, dP \\ &+ \int_{a_i \leq a_{-i}, d_i+1 \leq a_{-i}} a_{-i} - d_i \, dP + \int_{a_{-i} \leq a_i, a_i \leq z_{-i}} a_i - d_i \, dP = \\ &\int_{z_{-i}}^{d_i+1} (z_i - d_i)(x - d_{-i}) \, dx + \int_{d_{-i}}^{d_i+1} \int_{d_i}^{x} x - d_i \, dy \, dx + \\ &\int_{d_i+1}^{d_{-i}+1} \int_{d_i}^{d_i+1} x - d_i \, dy \, dx + \int_{d_{-i}}^{z_{-i}} (x - d_i)(x - d_{-i}) \, dx = \\ &\int_{z_{-i}}^{d_i+1} (z_i - d_i)(x - d_{-i}) \, dx + \int_{d_{-i}}^{d_i+1} (x - d_i)^2 \, dx + \\ &\int_{d_i+1}^{d_{-i}+1} x - d_i \, dx + \int_{d_{-i}}^{z_{-i}} (x - d_i)(x - d_{-i}) \, dx = \\ &\int_{z_{-i}}^{d_i+1} (z_i - d_i)(x - d_{-i}) \, dx + \int_{d_{-i}-d_i}^{1} x^2 \, dx + \int_{1}^{d_{-i}+1-d_i} x \, dx + \int_{d_{-i}}^{z_{-i}} (x - d_i)(x - d_{-i}) \, dx \end{aligned}$$

Using the Leibniz integral rule, I now find derivatives for the case where, in addition to the conditions above, $d < d_{-i}$ also holds.

$$\frac{\partial E(c_i)}{\partial d_i} = (z_i - d_i)(d_i + 1 - d_{-i}) + \int_{z_{-i}}^{d_i+1} -(x - d_{-i})\,dx +$$
$$(d_{-i} + d_i)^2 - d_{-i} - 1 + d_i + \int_{d_{-i}}^{z_{-i}} -(x - d_{-i})\,dx =$$
$$(z_i - d_i)(d_i + 1 - d_{-i}) + (d_{-i} + d_i)^2 - d_{-i} - 1 + d_i + \int_{d_{-i}}^{d_i+1} -(x - d_{-i})\,dx =$$
$$(z_i - d_i)(d_i + 1 - d_{-i}) + (d_{-i} + d_i)^2 - d_{-i} - 1 + d_i - \int_{0}^{d_i+1-d_{-i}} x\,dx =$$
$$(z_i - d_i)(d_i + 1 - d_{-i}) + (d_{-i} + d_i)^2 - d_{-i} - 1 + d_i - \frac{(d_i + 1 - d_{-i})^2}{2}$$

When I apply $z_i = d_{-i} + 1$, I get the following.

$$\frac{\partial E(c_i)}{\partial d_i} = -\frac{(d_i - d_{-i})^2 + 1}{2} \tag{28}$$

$$\frac{\partial^2 E(c_i)}{\partial d_i^2} = d_{-i} - d_i \tag{29}$$

Using equations 26, 27, 28 and 29, I can state the derivatives for player i's utility when $d < d_{-i}$.

$$\frac{\partial E(u_i)}{\partial d_i} = (d_{-i} - d_i - 1)\bar{m}_i + \frac{(d_i - d_{-i})^2 + 1}{2} \tag{30}$$

$$\frac{\partial^2 E(u_i)}{\partial d_i^2} = -\bar{m}_i + d_i - d_{-i} < 0 \tag{31}$$

In formula 31, the inequality is by $d_i \le d_{-i}$. In equation 30, note that if $d_i + 1 = d_{-i}$, $\frac{\partial E(u_i)}{\partial d_i} > 0$. In other words, if $d_i + 1 = d_{-i}$, d_i is not optimal for any $\bar{m}_i$. There is no pure strategy Nash equilibrium with $d_i + 1 = d_{-i}$.

If $d = d_{-i}$, $d_i = d$ is the optimal departure time for $d_i = [d, d_{-i}]$. If $d < d_{-i}$, since the second order derivative is negative by formula 31, $d_i = d$ is the optimal departure time for $d_i = [d, d_{-i}]$ if and only if $\frac{\partial E(u_i)}{\partial d_i}(d) \le 0$. Therefore the following is the required condition for both the $d = d_{-i}$ case and the $d < d_{-i}$ case.

$$\bar{m}_i \ge \frac{(d_i - d_{-i})^2 + 1}{2(d_i + 1 - d_{-i})}$$

Now, I will move on to player -i. Fix the value of d_{-i} in the Nash equilibrium as d'. By lemma 7, $d_{-i} < d'$ is not optimal. If $d' = d_i$ and $d_{-i} \ge d_i + 1$, the meeting probability is 0. If $d' > d_i$, by lemma 18, $d_{-i} > d_i + 1$ is not optimal. Therefore, I only need consider $d_{-i} \in [d', d_i + 1]$. By formulas 21 and 22, when $d_{-i} \in [d', d_i + 1]$, $\zeta_{-i} = d_i + 1$ is optimal. I will find player -i's meeting chance, cost and if they exist, their derivatives under $d' \le d_{-i} \le d_i + 1$, $z_i = d' + 1$ and $\zeta_{-i} = d_i + 1$.

By lemma 2 and formula 10, the following 2 statements hold. For any $a_{-i} \in [d', z_i]$, the probability of meeting is 1. For any $a_{-i} > z_i$, the probability of meeting is 0. Using these 2 statements I derive the following.

$$E(M) = z_i - d_{-i} \tag{32}$$

If $d' < d_i + 1$, the following derivative exists.

$$\frac{\partial E(M)}{\partial d_{-i}} = -1 \tag{33}$$

$$E(c_{-i}) = E(r_{-i}) + E(w_{-i}) =$$
$$\frac{1}{2} + \int_{d_{-i} \le a_i \le d_i+1, d_{-i} \le a_{-i} \le d_i+1} max\{0, a_i - a_{-i}\}\, dP =$$
$$\frac{1}{2} + \int_{d_{-i}}^{d_i+1} \int_{x}^{d_i+1} y - x\, dy\, dx = \frac{1}{2} + \frac{(d_i + 1 - d_{-i})^3}{6} \tag{34}$$

If $d' < d_i + 1$, the following derivative exists.

$$\frac{\partial E(c_{-i})}{\partial d_{-i}} = -\frac{(d_i + 1 - d_{-i})^2}{2} \tag{35}$$

Recall that $z_i = d' + 1$ here. By equations 32 and 34, in order for player -i to weakly prefer coming to the meeting, the following condition needs to be fulfilled.

$$\bar{m}_{-i} \ge \frac{1}{2} + \frac{(d_i + 1 - d_{-i})^3}{6} \tag{36}$$

If $d' < d_i + 1$, I have the following derivative by equations 33 and 35.

$$\frac{\partial E(u_i)}{\partial d_{-i}} = -\bar{m}_{-i} + \frac{(d_i + 1 - d_{-i})^2}{2}$$

If formula 36 is fulfilled, $\bar{m}_i \ge \frac{1}{2}$ and since $d' \ge d_i$, the following holds.

$$\forall d_{-i} \in [d', d_i + 1] : \bar{m}_{-i} \ge \frac{(d_i + 1 - d_{-i})^2}{2}$$

In this case, player -i does not prefer to delay her departure. ■

Proof of Proposition 1.

Compare propositions 1 and 7. (1)∼(3) from both propositions map to each other in order. Ignoring 0 probability events and ζ_j for cases where the player j has a 0 probability to wait, (4)∼(6) of proposition 7 means (4) of proposition 1. ■

Proposition 8. *In the Nash equilibria of proposition 1, the following properties hold.*

$$\forall i \in \{1,2\}, z_i \ge a_i : \frac{\partial c_i(a_i - d_i, z_i - a_i)}{\partial z_i} = 1 \tag{37}$$

$$\forall i \in \{1,2\}, a_i \in [d_i, d_i + 1] : \frac{\gamma_{-i}(z_i)}{P(z_{-i} < a_i) + 1 - \Gamma_{-i}(z_i)} = \frac{\gamma_{-i}(z_i)}{1 - \Gamma_{-i}(z_i)}. \tag{38}$$

$$\forall a_2 \in [d_2, d_2 + 1] : \frac{\gamma_1(z_2)}{1 - \Gamma_1(z_2)} = \begin{cases} 0 & z_2 \in [d_2, d_1) \\ \frac{1}{d_1 + 1 - z_2} & z_2 \in [d_1, d_1 + 1) \\ \textit{does not exist.} & z_2 \ge d_1 + 1 \end{cases} \tag{39}$$

If $z_2 = d_1+1$, $\gamma_1(z_2) = 1$. *If* $z_2 > d_1+1$, $\gamma_1(z_2) = 0$.

$$\forall a_1 \in [d_1, d_1+1] : \frac{\gamma_2(z_1)}{1-\Gamma_2(z_1)} = \begin{cases} \frac{1}{d_2+1-z_1} & z_1 \in [d_1, d_2+1) \\ \textit{does not exist.} & z_1 \geq d_2+1 \end{cases} \tag{40}$$

If $z_1 = d_2+1$, $\gamma_2(z_1) = 1$. *If* $z_1 > d_2+1$, $\gamma_2(z_1) = 0$.

$$\frac{\partial E(M)}{\partial d_2} = \begin{cases} 0 & d_2 < \zeta_1 - 1 \\ d_1 - d_2 - 1 & d_2 \in (\zeta_1 - 1, d_1] \\ -1 & d_2 \in [d_1, \zeta_1) \\ d_2 - d_1 - 1 & d_2 \in (\zeta_1, d_1+1] \\ 0 & d_2 \geq d_1 + 1 \end{cases} \tag{41}$$

$$\frac{\partial E(c_2)}{\partial d_2} = \begin{cases} -1 & d_2 \leq d_1 - 1 \\ \frac{(d_2+1-d_1)^2}{2} - 1 & d_2 \in [d_1 - 1, \zeta_1 - 1) \\ -\frac{(d_1-d_2)^2+1}{2} & d_2 \in (\zeta_1 - 1, d_1] \\ -\frac{(d_1+1-d_2)^2}{2} & d_2 \in [d_1, \zeta_1) \\ \frac{(d_1-d_2)^2-1}{2} & d_2 \in (\zeta_1, d_1+1] \\ 0 & d_2 \geq d_1 + 1 \end{cases} \tag{42}$$

$$\frac{\partial E(M)}{\partial d_1} = \begin{cases} 0 & d_1 < \zeta_2 - 1 \\ -1 & d_1 \in (\zeta_2 - 1, \zeta_2) \\ 0 & d_1 > \zeta_2 \end{cases} \tag{43}$$

$$\frac{\partial E(c_1)}{\partial d_1} = \begin{cases} -1 & d_1 \leq d_2 - 1 \\ \frac{(d_1+1-d_2)^2}{2} - 1 & d_1 \in [d_2 - 1, d_2] \\ -\frac{(d_2+1-d_1)^2}{2} & d_1 \in [d_2, d_2+1] \\ 0 & d_1 \geq d_2 + 1 \end{cases} \tag{44}$$

Proof. These are pure strategy Nash equilibria by proposition 7. I will refer to the proof of proposition 7 throughout this proof.

$$\forall i : c_i(a_i - d_i, z_i - a_i) = z_i - d_i$$

Therefore, formula 37 holds.

If $a_2 \in [d_2, d_2+1]$, $P(z_1 < a_2) = 0$. If $a_1 \in [d_1, d_1+1]$, $P(z_2 < a_1) = 0$. Therefore, formula 38 holds.

For the rest of the proof, I will deal with the derivatives with respect to d_1 or d_2. I start at the Nash equilibrium. First, I will solve for the derivatives with respect to d_2 while leaving player 1's strategy and $\zeta_2 = d_1+1$ fixed.

When the domain is $d_2 \leq \zeta_1 - 1$, I have the following equation by lemma 2 and formula 10

$$E(M) = 1$$

$$\frac{\partial E(M)}{\partial d_2} = 0$$

I already have the derivatives for the case when the domain is $d_2 \in [\zeta_1 - 1, d_1]$ and it is a proper interval. I can use formula 26 and 28.

$$\frac{\partial E(M)}{\partial d_2} = d_1 - d_2 - 1$$

$$\frac{\partial E(c_2)}{\partial d_2} = -\frac{(d_2 - d_1)^2 + 1}{2}$$

For the case where the domain is $d_2 \in [d_1, \zeta_1]$, I have the following equations.

$$E(M) = \int_{d_2}^{d_1+1} x - d_2\, dx + \int_{d_2}^{\zeta_1} x - d_1\, dx = \int_0^{d_1+1-d_2} x\, dx + \int_{d_2}^{\zeta_1} x - d_1\, dx$$

$$\frac{\partial E(M)}{\partial d_2} = -(d_1 + 1 - d_2) - (d_2 - d_1) = -1$$

$$\begin{aligned}
&E(c_2) = \int_{d_1+1 \le a_2 \le d_2+1} a_2 - d_2\, dP + \int_{a_1 \le a_2, a_2 \le \zeta_1} a_2 - d_2\, dP + \\
&\int_{a_2 \le a_1} a_1 - d_2\, dP + \int_{a_1 \le a_2, \zeta_1 \le a_1 \le d_1+1} d_1 + 1 - d_2\, dP = \\
&\int_{d_1+1}^{d_2+1} x - d_2\, dx + \int_{d_2}^{\zeta_1} (x - d_2)(x - d_1)\, dx + \\
&\int_{d_2}^{d_1+1} \int_{d_2}^{x} x - d_2\, dy\, dx + \int_{\zeta_1}^{d_1+1} (d_1 + 1 - d_2)(x - d_1)\, dx = \\
&\int_{d_1+1-d_2}^{1} x\, dx + \int_{d_2}^{\zeta_1} (x - d_2)(x - d_1)\, dx + \\
&\int_{d_2}^{d_1+1} (x - d_2)^2\, dx + \int_{\zeta_1}^{d_1+1} (d_1 + 1 - d_2)(x - d_1)\, dx = \\
&\int_{d_1+1-d_2}^{1} x\, dx + \int_{d_2}^{\zeta_1} (x - d_2)(x - d_1)\, dx + \\
&\qquad\qquad \int_0^{d_1+1-d_2} x^2\, dx + \int_{\zeta_1}^{d_1+1} (d_1 + 1 - d_2)(x - d_1)\, dx
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial E(c_2)}{\partial d_2} = \\
&d_1 + 1 - d_2 + \int_{d_2}^{\zeta_1} -(x - d_1)\, dx - (d_1 + 1 - d_2)^2 + \int_{\zeta_1}^{d_1+1} -(x - d_1)\, dx = \\
&\qquad d_1 + 1 - d_2 - (d_1 + 1 - d_2)^2 - \int_{d_2}^{d_1+1} x - d_1\, dx = -\frac{(d_1 + 1 - d_2)^2}{2}
\end{aligned}$$

If the domain is $d_2 \in [\zeta_1, d_1 + 1]$ and it is a proper interval, by lemma 2 and formula 8, the players meet if and only if $a_1 \le a_2 \le d_1 + 1$.

$$E(M) = \frac{(d_1 + 1 - d_2)^2}{2}$$

$$\frac{\partial E(M)}{\partial d_2} = d_2 - d_1 - 1$$

$$E(c_2) =$$
$$\int_{d_1+1}^{d_2+1} x - d_2\, dx + \int_{d_2}^{d_1+1} \int_{d_2}^{x} x - d_2\, dy\, dx + \int_{d_2}^{d_1+1} (d_1 + 1 - d_2)(x - d_1)\, dx =$$
$$\int_{d_1+1-d_2}^{1} x\, dx + \int_{d_2}^{d_1+1} (x - d_2)^2\, dx + \int_{d_2}^{d_1+1} (d_1 + 1 - d_2)(x - d_1)\, dx =$$
$$\int_{d_1+1-d_2}^{1} x\, dx + \int_{0}^{d_1+1-d_2} x^2\, dx + \int_{d_2}^{d_1+1} (d_1 + 1 - d_2)(x - d_1)\, dx$$

$$\frac{\partial E(c_2)}{\partial d_2} =$$
$$d_1 + 1 - d_2 - (d_1 + 1 - d_2)^2 - (d_1 + 1 - d_2)(d_2 - d_1) + \int_{d_2}^{d_1+1} -(x - d_1)\, dx =$$
$$-\frac{1}{2} + \frac{(d_1 - d_2)^2}{2}$$

If the domain is $d_2 \geq d_1 + 1$, $E(M) = 0$ and $E(c_2) = 0.5$.

$$\frac{\partial E(M)}{\partial d_2} = \frac{\partial E(c_2)}{\partial d_2} = 0$$

If the domain is $d_2 \leq d_1 - 1$,

$$E(c_2) = \int a_1 - a_2\, dP = \int_{d_2}^{d_2+1} \int_{d_1}^{d_1+1} y - x\, dy\, dx = d_1 - d_2$$

and

$$\frac{\partial E(c_2)}{\partial d_2} = -1.$$

If the domain is $d_2 \in [d_1 - 1, \zeta_1 - 1]$, player 2's cost is the same as the case where $d_2 \leq d_1 \leq d_2 + 1$, $\zeta_1 = d_2 + 1$ and $\zeta_2 = d_1 + 1$. Therefore, I can use proposition 7's formula 24.

$$\frac{\partial E(c_2)}{\partial d_2} = \frac{(d_2 + 1 - d_1)^2}{2} - 1$$

Next, I will solve for the derivatives with respect to d_1 while leaving player 2's strategy and $\zeta_1 = d_2 + 1$ fixed. When the domain is $d_1 \leq \zeta_2 - 1$, I have the following equation by 2 and formula 10.

$$E(M) = 1$$

$$\frac{\partial E(M)}{\partial d_1} = 0$$

I already have the derivatives for the case when the domain is $d_1 \in [\zeta_2 - 1, d_2 + 1]$ and it is a proper interval, I can use proposition 7's formulas 33 and 35.

$$\frac{\partial E(M)}{\partial d_{-i}} = -1$$

$$\frac{\partial E(c_1)}{\partial d_1} = -\frac{(d_2 + 1 - d_1)^2}{2}$$

When the domain is $d_1 \in [d_2+1, \zeta_2]$ and it is a proper interval, $E(M) = \zeta_2 - d_1$ by lemma 2 and formula 8.

$$\frac{\partial E(M)}{\partial d_1} = -1$$

When the domain is $d_1 \geq \zeta_2$, $E(M) = 0$ by lemma 2 and formula 8.

$$\frac{\partial E(M)}{\partial d_1} = 0$$

If the domain is $d_1 \leq d_2 - 1$,

$$E(c_1) = \int a_2 - a_1 \, dP = \int_{d_1}^{d_1+1} \int_{d_2}^{d_2+1} y - x \, dy \, dx = d_2 - d_1$$

and

$$\frac{\partial E(c_1)}{\partial d_1} = -1.$$

When the domain is $d_1 \in [d_2-1, d_2]$, player 1's cost is the same as the case where $d_1 \leq d_2 \leq d_1+1$, $\zeta_1 = d_2+1$ and $\zeta_2 = d_1+1$. Therefore, I can use proposition 7's formula 24.

$$\frac{\partial E(c_1)}{\partial d_1} = \frac{(d_1+1-d_2)^2}{2} - 1$$

When the domain is $d_1 \geq d_2+1$,

$$E(c_1) = 0.5$$

and

$$\frac{\partial E(c_1)}{\partial d_1} = 0.$$

$\blacksquare$

Proof of Proposition 2.

In this proof, for the purpose of taking derivatives, $d_1 \geq 1$ and d_2 is a variable whose domain is $d_2 \in (d_1-1, d_1]$. I will first prove m_2' and m_2'' are increasing in $d_1 - d_2$.

$$\frac{\partial m_2'}{\partial d_2} = \frac{(d_2+1-d_1)^2}{2} - 1 < 0 \tag{45}$$

$$\frac{\partial m_2''}{\partial d_2} = \frac{4(d_2-d_1)(d_2+1-d_1) - 2(d_2-d_1)^2 - 2}{4(d_2+1-d_1)^2} < 0$$

Next, I will prove m_1' is decreasing in $d_1 - d_2$.

$$\frac{\partial m_1'}{\partial d_2} = \frac{(d_2+1-d_1)^2}{2} > 0 \tag{46}$$

Finally, I prove I will first prove $m_1' + m_2'$ is increasing in $d_1 - d_2$. Combine formulas 45 and 46.

$$\frac{\partial (m_1' + m_2')}{\partial d_2} = (d_2+1-d_1)^2 - 1 \leq 0$$

When $d_2 \in (d_1-1, d_1)$

$$\frac{\partial (m_1' + m_2')}{\partial d_2} < 0$$

$\blacksquare$

Appendix 3. Results and proofs used in subsection 4.2

This appendix section contains a proposition, lemmas and a example used in subsection 4.2. The proof of propositions 3∼5 is also written here. Lemmas used in this section but not found in this paper are in chapter 2 of the Supplemental Material.

Lemma 4. *The following holds under assumptions 3 and 4.*

(1)

$$P(a_i \le x) = \begin{cases} 0 & x \le \underline{s} \\ \frac{x^2-\underline{s}^2}{2} & x \in [\underline{s}, \bar{s}] \\ \bar{s}x - \frac{\bar{s}^2+\underline{s}^2}{2} & x \in [\bar{s}, \underline{s}+1] \\ \bar{s} - \frac{(\bar{s}+1-x)^2}{2} & x \in [\underline{s}+1, \bar{s}+1] \\ \bar{s} & x \ge \bar{s}+1 \end{cases}$$

(2)

$$\frac{\partial P(a_i \le x)}{\partial x} = \begin{cases} 0 & x < \underline{s} \\ x & x \in (\underline{s}, \bar{s}] \\ \bar{s} & x \in [\bar{s}, \underline{s}+1) \\ \bar{s}+1-x & x \in (\underline{s}+1, \bar{s}+1] \\ 0 & x \ge \bar{s}+1 \end{cases}$$

Proof. I will prove (1) first. (2) follows from (1).

Player i never departs before $\underline{s}$. If $a \le \underline{s}$, $P(a_i \le a) = 0$. If $\underline{s} \le a \le \bar{s}$,

$$P(a_i \le a) = \underline{s}(a-\underline{s}) + \int_{\underline{s}}^{a}\int_{x}^{a} dy\,dx = \underline{s}(a-\underline{s}) + \int_{\underline{s}}^{a} a - x\,dx = \underline{s}(a-\underline{s}) + \frac{(a-\underline{s})^2}{2} = \frac{a^2-\underline{s}^2}{2}.$$

If $\bar{s} \le a \le \underline{s}+1$,

$$P(a_i \le a) = \underline{s}(a-\underline{s}) + \int_{\underline{s}}^{\bar{s}}\int_{x}^{a} dy\,dx = \underline{s}(a-\underline{s}) + \int_{\underline{s}}^{\bar{s}} a - x\,dx = \underline{s}(a-\underline{s}) + a(\bar{s}-\underline{s}) - \frac{\bar{s}^2}{2} + \frac{\underline{s}^2}{2} = \bar{s}a - \frac{\bar{s}^2+\underline{s}^2}{2}.$$

If $\underline{s}+1 \le a \le \bar{s}+1$,

$$P(a_i \le a) = \underline{s} + \int_{\underline{s}}^{a-1} dx + \int_{a-1}^{\bar{s}}\int_{x}^{a} dy\,dx =$$
$$\underline{s} + a - 1 - \underline{s} + \int_{a-1}^{\bar{s}} a - x\,dx =$$
$$= \underline{s} + a - 1 - \underline{s} + a(\bar{s}+1-a) - \frac{\bar{s}^2}{2} + \frac{(a-1)^2}{2} = \bar{s} - \frac{(\bar{s}+1-a)^2}{2}.$$

Player i's probability of departure is $\bar{s}$. If player i never departs for the meeting place. She always arrives before or at $\bar{s}+1$. If $a \ge \bar{s}+1$, $P(a_i \le a) = \bar{s}$. ■

The above lemma deals with the distribution of player i's arrival time, a_i. It states $P(a_i \leq x)$ and $\frac{\partial P(a_i \leq x)}{\partial x}$. Definition 3 uses these to create definition 3's CDF and PDF.

Proposition 9. *Under assumption 3 for both players and assumption 4 for player $-i$, the following formulas hold for player i when $\zeta_i = \underline{s}+1$.*

(1)

$$E(M|d_i) = \begin{cases} \bar{s} - \frac{(\bar{s}-\underline{s})^2}{2} & d_i \leq \underline{s} \\ (\bar{s} - \frac{(\bar{s}-\underline{s})^2}{2})(\underline{s}+1-d_i) & d_i \in [\underline{s}, \underline{s}+1] \\ 0 & d_i > \underline{s}+1 \end{cases}$$

(2) If $d_i \leq \underline{s}$,

$$E(w_i|d_i) = \int_{d_i}^{\underline{s}} \int_{\underline{s}}^{\bar{s}} (y-x) y\, dy\, dx + \int_{d_i}^{\underline{s}} \int_{\bar{s}}^{\underline{s}+1} (y-x)\bar{s}\, dy\, dx +$$
$$\int_{\underline{s}}^{\bar{s}} \int_{x}^{\bar{s}} (y-x) y\, dy\, dx + \int_{\underline{s}}^{\bar{s}} \int_{\bar{s}}^{\underline{s}+1} (y-x)\bar{s}\, dy\, dx +$$
$$\int_{\bar{s}}^{d_i+1} \frac{\underline{s}+1-x}{2} \bar{s}(\underline{s}+1-x)\, dx + (1-\bar{s}+\frac{(\bar{s}-\underline{s})^2}{2})(\underline{s}-d_i+0.5).$$

If $d_i \in [\underline{s}, \bar{s}]$,

$$E(w_i|d_i) = \int_{d_i}^{\bar{s}} \int_{x}^{\bar{s}} (y-x) y\, dy\, dx + \int_{d_i}^{\bar{s}} \int_{\bar{s}}^{\underline{s}+1} (y-x)\bar{s}\, dy\, dx$$
$$+ \int_{\bar{s}}^{\underline{s}+1} \frac{\underline{s}+1-x}{2} \bar{s}(\underline{s}+1-x)\, dx \tag{47}$$
$$+ (1-\bar{s}+\frac{(\bar{s}-\underline{s})^2}{2})(\underline{s}+1-d_i)\frac{\underline{s}+1-d_i}{2}.$$

If $d_i \in [\bar{s}, \underline{s}+1]$,

$$E(w_i|d_i) =$$
$$\int_{d_i}^{\underline{s}+1} \frac{\underline{s}+1-x}{2} \bar{s}(\underline{s}+1-x)\, dx + (1-\bar{s}+\frac{(\bar{s}-\underline{s})^2}{2})(\underline{s}+1-d_i)\frac{\underline{s}+1-d_i}{2}. \tag{48}$$

If $d_i \geq \underline{s}+1$,

$$E(w_i|d_i) = 0.$$

Proof. (1) uses the fact that when $\zeta_i = \underline{s}+1$, by lemma 2 and formula 8, the players meet when $a_i \leq \underline{s}+1$ and $a_{-i} \leq \underline{s}+1$.

(2) uses the fact that when $x \in [\bar{s}, \underline{s}+1)$, $E(w_i|a_i = x, a_{-i} \in (x, \underline{s}+1]) = \frac{\underline{s}+1-x}{2}$. I also use $P(a_{-i} > \underline{s}+1) = (1-\bar{s}+\frac{(\bar{s}-\underline{s})^2}{2})$ and the fact that when $d_i \in [\underline{s}, \underline{s}+1]$, $P(a_i \leq \underline{s}+1) = (\underline{s}+1-d_i)$. ■

The above proposition states player i's expected meeting probability and expected wait cost for given values of her departure time, d_i. For this, I assume that player $-i$'s strategy follows assumption 4 and player i's planned abandonment time is $\zeta_i = \underline{s}+1$. Recall that $E(c_i) = E(r_i) + E(w_i) = 0.5 + E(w_i)$. Therefore, I can get the expected cost from the expected wait cost. This proposition allows me to analyze the expected utility of player i in the symmetric Nash equilibria of assumptions 3 and 4 at different departure times for her. In subsection 4.2's figures 4, I use this proposition to graph player i's expected benefit and cost of meeting.

Example 2. *Suppose player -i's arrival time follows definition 3 and that $\zeta_{-i} = \underline{s}+1$.*

(1) Under assumption 3, when d_i and a_i are given,

$$\frac{\partial c_i(a_i - d_i, z_i - a_i)}{\partial z_i} = 1.$$

(2) when a_i is given,

$$\frac{\gamma_{-i}(z_i)}{E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i) + 1 - \Gamma_{-i}(z_i)} = \frac{\gamma_{-i}(z_i)}{P(z_{-i} < a_i) + 1 - \Gamma_{-i}(z_i)}$$

(3) If $a_i \in [0, \underline{s}+1]$, $z_i \leq 8$ and $\bar{s} < 1$,

$$\frac{\gamma_{-i}(z_i)}{P(z_{-i} < a_i) + 1 - \Gamma_{-i}(z_i)} = \frac{\gamma_{-i}(z_i)}{1 - \Gamma_{-i}(z_i)}$$

and

$$\frac{\gamma_{-i}(z_i)}{1 - \Gamma_{-i}(z_i)} = \begin{cases} \frac{2z_i}{\underline{s}^2 - z_i^2 + 2} & z_i \in [\underline{s}, \bar{s}] \\ \frac{2\bar{s}}{-2\bar{s}z_i + \bar{s}^2 + \underline{s}^2 + 2} & z_i \in [\bar{s}, \underline{s}+1] \\ \frac{2(\bar{s}+1-z_i)}{(\bar{s}+1-z_i)^2 - 2\bar{s} + 2} & z_i \in (\underline{s}+1, \bar{s}+1] \\ 0 & \textit{otherwise} \end{cases}$$

(4) If $\underline{s}+1 < a_i$ and $z_i \leq \bar{s}+1$,

$$\frac{\gamma_{-i}(z_i)}{P(z_{-i} < a_i) + 1 - \Gamma_{-i}(z_i)} = \frac{2(\bar{s}+1-z_i)}{(\bar{s}+1-z_i)^2 + 2 - (\bar{s}+1-a_i)^2} < \frac{2(\bar{s}+1-z_i)}{(\bar{s}+1-z_i)^2 - 2\bar{s} + 2}$$

(5) If $\bar{s}+1 \leq z_i \leq 8$ and $\bar{s} < 1$,

$$\frac{\gamma_{-i}(z_i)}{P(z_{-i} < a_i) + 1 - \Gamma_{-i}(z_i)} = 0$$

The above example assumes that player -i's arrival time follows definition 3's CDF and PDF and that $\zeta_{-i} = \underline{s}+1$ to solve $\frac{\partial c_i(a_i - d_i, z_i - a_i)}{\partial z_i}$ and $\frac{\gamma_{-i}(z_i)}{E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i)+1-\Gamma_{-i}(z_i)}$. This means that the example solves for the two functions about player i when player -i's strategy is almost the same as her strategy when player -i follows assumptions 3 and 4. The two functions about player i for are the key to performing hazard rate analysis in subsection 4.2. The example often simplifies $\frac{\gamma_{-i}(z_i)}{E(\mathbb{1}_{P(z_{-i}<a_i)}|a_i)+1-\Gamma_{-i}(z_i)}$ to $\frac{\gamma_{-i}(z_i)}{P(z_{-i}<a_i)+1-\Gamma_{-i}(z_i)}$ or $\frac{\gamma_{-i}(z_i)}{1-\Gamma_{-i}(z_i)}$ when it can.

Proof of Proposition 4.

Lemma 21's (1) and lemma 25 establish the necessary and sufficient conditions under which assumption 4 is a Nash equilibrium. Consider the case where $\bar{s} = 1$. In this case, by lemma 26, there exists no $\bar{m}_{-i}$ fulfilling the optimality condition of lemma 25. I need only consider the case where $\bar{s} < 1$.

When $\bar{s} < 1$, by lemma 28, $\bar{m}_i > \frac{-(\bar{s}-\underline{s})^3 + 3(\bar{s}-\underline{s})^2 - 3\bar{s} + 6}{6\bar{s} - 3(\bar{s}-\underline{s})^2}$ is implied by $\bar{m}_i = \bar{\iota}(\underline{s}, \bar{s})$.

Next, I will prove that if $\hat{s}$ exists, $\hat{s} - \underline{s} < \frac{1}{3}$.

$$\frac{d(\frac{1}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2})}{d(\bar{s}-\underline{s})} = -\frac{1}{(\bar{s}-\underline{s})^2} + \frac{1}{2} < 0 \tag{49}$$

$\bar{w}(\underline{s},\bar{s}) = \frac{1}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2}$ is decreasing in $\bar{s}-\underline{s}$.

$$\frac{\partial(2(\bar{s}-\underline{s}) - (\bar{s}-\underline{s})^2)}{\partial(\bar{s}-\underline{s})} = \underline{s} + 1 - \bar{s} > 0 \tag{50}$$

$2(\bar{s}-\underline{s}) - (\bar{s}-\underline{s})^2$ is increasing in $\bar{s}-\underline{s}$.

Consider the case where $\bar{s}-\underline{s} \geq 0.5$. This implies $\bar{s} \geq 0.5$.

$$\begin{gathered}\bar{i}(\underline{s},\bar{s}) = \frac{6 + 2(\bar{s}+3)(\underline{s}+1-\bar{s})^3 + 3((\bar{s}-\underline{s})^2 - 2\underline{s})(\underline{s}+1-\bar{s})^2}{(12\bar{s} - 6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} \geq \\ \frac{6}{(12\bar{s} - 6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} \geq \frac{12}{12\bar{s} - 6(\bar{s}-\underline{s})^2} \geq \frac{1}{\bar{s}-\frac{1}{8}} \geq \frac{8}{7}\end{gathered} \tag{51}$$

Here, the first weak inequality uses equation 34.

$$\bar{w}(\underline{s},\bar{s}) = \frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2} = \frac{1-\bar{s}+\underline{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2} - \frac{\underline{s}}{\bar{s}-\underline{s}} = \frac{1}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2} - \frac{\underline{s}}{\bar{s}-\underline{s}} - 1 \tag{52}$$

By formula 49, I have the following.

$$\bar{w}(\underline{s},\bar{s}) \leq 2 + \frac{1}{4} - \frac{\underline{s}}{\bar{s}-\underline{s}} - 1 \leq \frac{5}{4} - \underline{s} \tag{53}$$

When $\underline{s} \geq \frac{1}{4}$, formulas 51 and 53 mean that this is not a Nash equilibrium. Suppose $\underline{s} < \frac{1}{4}$.

$$12\bar{s} - 6(\bar{s}-\underline{s})^2 = 6(2\bar{s} - (\bar{s}-\underline{s})^2) = 6(2(\bar{s}-\underline{s}) + 2\underline{s} - (\bar{s}-\underline{s})^2) < 6 + 12\underline{s} < 9$$

Combine the above result with $\frac{12}{12\bar{s}-6(\bar{s}-\underline{s})^2}$ from formula 51.

$$\bar{i}(\underline{s},\bar{s}) > \frac{4}{3} \tag{54}$$

Formula 53 and inequality 54 means that this is not a Nash equilibrium.

Consider the case where $\frac{1}{3} \leq \bar{s}-\underline{s} < 0.5$ and $\underline{s} \leq \frac{5}{11}$.

$$\frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2} \leq 3 - 3\bar{s} + \frac{1}{4} \tag{55}$$

$$\begin{gathered}\frac{6 + 2(\bar{s}+3)(\underline{s}+1-\bar{s})^3 + 3((\bar{s}-\underline{s})^2 - 2\underline{s})(\underline{s}+1-\bar{s})^2}{(12\bar{s} - 6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} \geq \\ \frac{6}{(12\bar{s} - 6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} + \frac{(6 - 6\bar{s} + 3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2}{(12\bar{s} - 6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} + \\ \frac{2\bar{s}(\underline{s}+1-\bar{s})^3}{(12\bar{s} - 6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} \geq \\ \frac{6}{(12\bar{s}-\frac{2}{3})(\frac{2}{3})} + \frac{(6 - 6\bar{s} + 3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})}{12\bar{s} - 6(\bar{s}-\underline{s})^2} + \frac{2\bar{s}(\underline{s}+1-\bar{s})^2}{12\bar{s} - 6(\bar{s}-\underline{s})^2} > \\ \frac{9}{12\bar{s}-\frac{2}{3}} + \frac{(6 - 6(\bar{s}-\underline{s}) + 3(\bar{s}-\underline{s})^2 - 6\underline{s})(\underline{s}+1-\bar{s})}{12(\bar{s}-\underline{s}) - 6(\bar{s}-\underline{s})^2 + 12\underline{s}} + \frac{(\underline{s}+1-\bar{s})^2}{6} \geq \\ \frac{9}{12\bar{s}-\frac{2}{3}} + \frac{(\frac{15}{4} - 6\underline{s})(\frac{1}{2})}{\frac{9}{2} + 12\underline{s}} + \frac{1}{24} \geq \frac{9}{12\bar{s}-\frac{2}{3}} + \frac{45}{876} + \frac{1}{24} > \frac{9}{12\bar{s}-\frac{2}{3}} + \frac{1}{11}\end{gathered} \tag{56}$$

Here, the first weak inequality uses equation 34. The penultimate weak inequality uses the fact that $\frac{6-3x-6\underline{s}}{6x+12\underline{s}}$ is decreasing in x and formula 50. Combine formulas 55 and 56. In a Nash equilibrium, the following holds.

$$\frac{13}{4}-3\bar{s} > \frac{9}{12\bar{s}-\frac{2}{3}}+\frac{1}{11}$$

$$(\frac{13}{4}-3\bar{s})(12\bar{s}-\frac{2}{3}) > 9+\frac{1}{11}(12\bar{s}-\frac{2}{3})$$

The above formula has no real roots. This is not a Nash equilibrium.

Consider the case where $\frac{1}{3} \le \bar{s}-\underline{s} < 0.5$ and $\underline{s} > \frac{5}{11}$. From formulas 49 and 52, $\frac{1}{\bar{s}-\underline{s}}+\frac{\bar{s}-\underline{s}}{2}$ is decreasing in $\bar{s}-\underline{s}$.

$$\frac{1-\bar{s}}{\bar{s}-\underline{s}}+\frac{\bar{s}-\underline{s}}{2} \le 3+\frac{1}{6}-\frac{\underline{s}}{\bar{s}-\underline{s}}-1 \le \frac{13}{6}-3\underline{s} \tag{57}$$

By equation 34, I have the following.

$$\frac{6+2(\bar{s}+3)(\underline{s}+1-\bar{s})^3+3((\bar{s}-\underline{s})^2-2\underline{s})(\underline{s}+1-\bar{s})^2}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} \ge \frac{6}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} \tag{58}$$

$$\frac{\partial((12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s}))}{\partial\bar{s}} = 12(\underline{s}+1-\bar{s})^2-(12\bar{s}-6(\bar{s}-\underline{s})^2) =$$
$$12(\underline{s}+1-\bar{s})^2-(12(\bar{s}-\underline{s})+12\underline{s}-6(\bar{s}-\underline{s})^2) <$$
$$12\times\frac{4}{9}-(12\times\frac{1}{3}+12\times\frac{5}{11}-6\times\frac{1}{9}) < 0$$

The above inequality uses formula 50. $(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})$ is decreasing in $\bar{s}$

$$\frac{6+2(\bar{s}+3)(\underline{s}+1-\bar{s})^3+3((\bar{s}-\underline{s})^2-2\underline{s})(\underline{s}+1-\bar{s})^2}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} > \frac{6}{(12(\frac{1}{3}+\underline{s})-6\times\frac{1}{9})\times\frac{2}{3}} \tag{59}$$

By formulas 57 and 59, the following holds in a Nash equilibrium.

$$(\frac{13}{6}-3\underline{s})(\frac{10}{3}+12\underline{s}) > 9$$

$$\frac{d((\frac{13}{6}-3\underline{s})(\frac{10}{3}+12\underline{s}))}{d\underline{s}} = -72\underline{s}+16 < 0$$

$(\frac{13}{6}-3\underline{s})(\frac{10}{3}+12\underline{s})$ is decreasing in $\underline{s}$.

$$(\frac{13}{6}-3\times\frac{5}{11})(\frac{10}{3}+12\times\frac{5}{11}) < 9$$

This is not a Nash equilibrium. Since $\bar{i}(\underline{s},\bar{s}) \le \bar{w}(\underline{s},\bar{s})$ does not hold when $\bar{s}-\underline{s} \ge \frac{1}{3}$, Nash equilibria satisfy $\bar{s}-\underline{s} < \frac{1}{3}$. Also, if $\hat{s}$ exists, $\hat{s}-\underline{s} < \frac{1}{3}$ holds in the Nash equilibria of the proposition.

Next, I will prove that when $\bar{i}(\underline{s},\bar{s}) \leq \bar{w}(\underline{s},\bar{s})$, $\bar{i}(\underline{s},\bar{s})$ is decreasing in $\bar{s}$. By the quotient rule and equation 34, it is sufficient to show that the following inequality holds.

$$\begin{aligned}&\frac{\partial((12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s}))}{\partial \bar{s}} \times \\ &\frac{6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} > \\ &\frac{\partial(6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2}{\partial \bar{s}}\end{aligned} \tag{60}$$

$$\begin{aligned}&\frac{\partial(6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2}{\partial \bar{s}} = \\ &-6(\underline{s}+1-\bar{s})^3-6\bar{s}(\underline{s}+1-\bar{s})^2-(12-12\bar{s}+6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})<0\end{aligned} \tag{61}$$

$$\frac{\partial((12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s}))}{\partial \bar{s}} = 12((\underline{s}+1-\bar{s})^2-(\bar{s}-\frac{(\bar{s}-\underline{s})^2}{2})) \tag{62}$$

Consider the case where $\bar{s}-\underline{s} \geq \frac{1}{4}$.

$$\begin{aligned}&\frac{12((\underline{s}+1-\bar{s})^2-(\bar{s}-\frac{(\bar{s}-\underline{s})^2}{2}))}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})} = \frac{2(\underline{s}+1-\bar{s})}{2\bar{s}-(\bar{s}-\underline{s})^2} - \frac{1}{\underline{s}+1-\bar{s}} \geq \\ &\frac{\frac{3}{2}}{\frac{1}{2}+2\underline{s}-\frac{1}{16}} - \frac{4}{3} \geq \frac{24}{39} - \frac{4}{3} = -\frac{28}{39}\end{aligned}$$

Here, the first weak inequality uses $\frac{\partial(2\bar{s}-(\bar{s}-\underline{s})^2)}{\partial \bar{s}} > 0$. Therefore, a sufficient condition is the following.

$$\begin{aligned}&-(6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2) \times \frac{28}{39} > \\ &-6(\underline{s}+1-\bar{s})^3-6\bar{s}(\underline{s}+1-\bar{s})^2-(12-12\bar{s}+6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})\end{aligned}$$

$$\begin{aligned}&6(\underline{s}+1-\bar{s})^3+6\bar{s}(\underline{s}+1-\bar{s})^2+(12-12\bar{s}+6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s}) > \\ &(6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2) \times \frac{28}{39}\end{aligned} \tag{63}$$

$$\frac{\partial(6(\underline{s}+1-\bar{s})^3+6(\bar{s}-\underline{s})^2(\underline{s}+1-\bar{s}))}{\partial(\bar{s}-\underline{s})} = -36(\bar{s}-\underline{s})^2+48(\bar{s}-\underline{s})-18<0 \tag{64}$$

$$\frac{\partial(12(\underline{s}+1-\bar{s})-\frac{28}{39}\times 6(\underline{s}+1-\bar{s})^2)}{\partial(\bar{s}-\underline{s})} < 0 \tag{65}$$

$$\forall y \geq 0, \frac{\partial(6\bar{s}(\underline{s}+1-\bar{s})^2-\frac{28}{39}(2\bar{s}(\underline{s}+1-\bar{s})^3-y(\underline{s}+1-\bar{s})^2))}{\partial(\bar{s}-\underline{s})} < 0 \tag{66}$$

Using formulas 64, 65 and 66, I transform inequality 63. For this, I can use $\underline{s}-\bar{s} < \frac{1}{3}$ since $\bar{i}(\underline{s},\bar{s}) \leq \bar{w}(\underline{s},\bar{s})$.

$$\begin{aligned}&6\times\frac{27}{64}+6\bar{s}\times\frac{9}{16}+(12-12\bar{s}+\frac{6}{16})\times\frac{3}{4} > \\ &\frac{28}{39}(6+2\bar{s}\times\frac{27}{64}+(6-6\bar{s}+\frac{3}{9})\times\frac{9}{16})\end{aligned}$$

I transform the above using $\frac{27}{8}-9-\frac{28}{39}(\frac{54}{64}-\frac{54}{16})<0$.

$$6\times\frac{27}{64}+6\times\frac{9}{16}+\frac{6}{16}\times\frac{3}{4}>\frac{28}{39}(6+2\times\frac{27}{64}+\frac{3}{9}\times\frac{9}{16})$$

Since the above inequality holds, the $\bar{s}-\underline{s}\geq\frac{1}{4}$ case is proven.

Now consider the case where $\bar{s}-\underline{s}<\frac{1}{4}$.

$$\frac{\partial((\underline{s}+1-\bar{s})^2+\frac{(\bar{s}-\underline{s})^2}{2})}{\partial(\bar{s}-\underline{s})}=3(\bar{s}-\underline{s})-2<0$$

Therefore,

$$(\underline{s}+1-\bar{s})^2-(\bar{s}-\frac{(\bar{s}-\underline{s})^2}{2})>\frac{19}{32}-\bar{s}.$$

If $\bar{s}\leq\frac{19}{32}$, by formulas 60, 61 and 62, the case is proven. If $\bar{s}>\frac{19}{32}$, formulas 60, 61 and 62 give me the following sufficient condition.

$$\begin{aligned}12(\frac{19}{32}-\bar{s})\frac{6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})}>&\\ -6(\underline{s}+1-\bar{s})^3-6\bar{s}(\underline{s}+1-\bar{s})^2-(12-12\bar{s}+6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})&\end{aligned}$$

$$\begin{aligned}6(\underline{s}+1-\bar{s})^3+6\bar{s}(\underline{s}+1-\bar{s})^2+(12-12\bar{s}+6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})>&\\ 12(\bar{s}-\frac{19}{32})\frac{6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2}{(12\bar{s}-6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})}&\end{aligned}$$

From the above, I use the following to transform the inequality.

$$\frac{\bar{s}-\frac{19}{32}}{\bar{s}-0.5(\bar{s}-\underline{s})^2}\leq\frac{\bar{s}-\frac{19}{32}}{\bar{s}-0.125}\leq\frac{\frac{13}{32}}{0.875}<0.5$$

$$\begin{aligned}12(\underline{s}+1-\bar{s})^4+12\bar{s}(\underline{s}+1-\bar{s})^3+2(12-12\bar{s}+6(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2>&\\ 6+2\bar{s}(\underline{s}+1-\bar{s})^3+(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2&\end{aligned}$$

$$12(\underline{s}+1-\bar{s})^4+10\bar{s}(\underline{s}+1-\bar{s})^3+3(6-6\bar{s}+3(\bar{s}-\underline{s})^2)(\underline{s}+1-\bar{s})^2>6$$

Since $12\times0.75^4+10\times\frac{19}{32}\times0.75^3>6$, the above holds. The case is proven.

Next, I prove that $\underline{s}>0$ is required for a Nash equilibrium. From definition 4, in the special case of $\underline{s}=0$, by lemma 27, $s_p(\bar{s})$ is positive. Therefore, by lemma 29, there is no Nash equilibrium in this case.

Now, consider the general case of $0<\underline{s}<1$. Lemma 30 gives $\hat{s}$. By lemma 29, this $\hat{s}$ satisfies $\bar{i}(\underline{s},\hat{s})=\bar{w}(\underline{s},\hat{s})$. Furthermore, by the same lemmas if $\bar{s}\in(\underline{s},\hat{s})$, $\bar{i}(\underline{s},\bar{s})<\bar{w}(\underline{s},\bar{s})$ and if $\bar{s}\in(\hat{s},1]$, $\bar{i}(\underline{s},\bar{s})>\bar{w}(\underline{s},\bar{s})$. ■

A large part of the proof of proposition 4 is handled by lemmas 20∼30. After stating proposition 4, subsection 4.2 explains the logic of the whole proof that goes through the lemmas. Disregarding the lemmas for now, most of the space in the above proof is spent on proving propositions 4.3 and 4.4. Both of these are proven by separating possible values of $\bar{s}-\underline{s}$ into different cases and proving for each case.

Proof of Proposition 3.

Necessity is is by proposition 4's (1) and (2). I will prove sufficiency. If $\bar{s} < 1$ and $\bar{m}_1 = \bar{m}_2 = \bar{i}(\underline{s}, \bar{s}) \leq \bar{w}(\underline{s}, \bar{s})$, by lemma 28, $\bar{m}_i > \frac{-(\bar{s}-\underline{s})^3+3(\bar{s}-\underline{s})^2-3\bar{s}+6}{6\bar{s}-3(\bar{s}-\underline{s})^2}$ is satisfied. Then, lemma 21's (1) and lemma 25 show that assumption 4 for both players is a Nash equilibrium. Lemma 26 establishes that if $\bar{s} = 1$, $\bar{i}(\underline{s}, \bar{s}) \leq \bar{w}(\underline{s}, \bar{s})$ is violated. ■

Lemma 5. *If $\bar{s} < 1$ and $\bar{s} - \underline{s} \geq \sqrt{2-2\bar{s}}$, $\bar{i}(\underline{s}, \bar{s}) > \bar{w}(\underline{s}, \bar{s})$*

Proof. Suppose $\bar{s} - \underline{s} = \sqrt{2-2\bar{s}+\varepsilon}$ for $\varepsilon \geq 0$. $\bar{s} \geq \sqrt{2-2\bar{s}+\varepsilon}$. From definition 4, I have the following.

$$s_p(\bar{s}) =$$
$$(\frac{1}{2} + \bar{s}\frac{(\underline{s}+1-\bar{s})^3}{6})\sqrt{2-2\bar{s}+\varepsilon} - (2-2\bar{s}+\frac{\varepsilon}{2})(1-\sqrt{2-2\bar{s}+\varepsilon})\frac{\bar{s}+\underline{s}}{2} \geq$$
$$(\frac{1}{2} + \bar{s}\frac{(\underline{s}+1-\bar{s})^3}{6})\sqrt{2-2\bar{s}+\varepsilon} - (2-2\bar{s}+\varepsilon)(1-\sqrt{2-2\bar{s}+\varepsilon})\frac{\bar{s}+\underline{s}}{2}$$

$$\frac{s_p(\bar{s})}{\sqrt{2-2\bar{s}+\varepsilon}} = \frac{1}{2} + \bar{s}\frac{(\underline{s}+1-\bar{s})^3}{6} - (\sqrt{2-2\bar{s}+\varepsilon} - (2-2\bar{s}+\varepsilon))\frac{\bar{s}+\underline{s}}{2} \geq$$
$$\frac{1}{2} + \bar{s}\frac{(\underline{s}+1-\bar{s})^3}{6} - \frac{\bar{s}+\underline{s}}{8} > 0$$

Therefore, by lemma 29, when $\bar{s} - \underline{s} \geq \sqrt{2-2\bar{s}}$, $\bar{i}(\underline{s}, \bar{s}) > \bar{w}(\underline{s}, \bar{s})$. ■

When $\bar{s} < 1$, $\bar{s} - \underline{s} \geq \sqrt{2-2\bar{s}}$ means $\bar{i}(\underline{s}, \bar{s}) > \bar{w}(\underline{s}, \bar{s})$. Here, $\bar{s} - \underline{s} \geq \sqrt{2-2\bar{s}}$ says that the difference of $\underline{s}$ and $\bar{s}$ is large or alternatively that $\underline{s}$ is small compared to $\bar{s}$.

Proof of Proposition 5.

Proposition 4.2 establishes that under assumption 3 for both players, Nash equilibria where assumption 4 holds for both players satisfies $0 < \underline{s}$ and $\bar{s} < 1$. This allows me to solve under the assumption that $\bar{s} < 1$.

In this proof, I allow deviating events for conditional expectations. I will denote player $-i$'s original strategy following assumption 4 as s_{-i}. Suppose that player $-i$ plays the following strategy, s'_{-i} instead. Define $s' \in (\underline{s}, \bar{s})$ and $\varepsilon = s' - \underline{s}$. If $s_{-i} \leq s'$, $d_{-i} = s'$. If $s_{-i} \in (s', \bar{s}]$, $d_{-i} = s_{-i}$. If $s_{-i} > \bar{s}$, player $-i$ does not depart for the meeting place. If player $-i$ departs, $\zeta_{-i} = s' + 1$.

By proposition 9's (1),

$$E(M|d_i = \bar{s}, \zeta_i = \underline{s}+1, s_{-i}) = (\bar{s} - \frac{(\bar{s}-\underline{s})^2}{2})(\underline{s}+1-\bar{s}).$$

By lemma 4,

$$E(M|d_i = \bar{s}+\varepsilon, \zeta_i = s'+1, s'_{-i}) = (\bar{s} - \frac{(\bar{s}-\underline{s}-\varepsilon)^2}{2})(\underline{s}+1-\bar{s}).$$

The above equation uses the fact that in this case, lemma 2 and formula 8, the players meet when $a_i \leq s'+1$ and $a_{-i} \leq s'+1$.

$$E(M|d_i = \bar{s}, \zeta_i = \underline{s}+1, s_{-i}) < E(M|d_i = \bar{s}+\varepsilon, \zeta_i = s'+1, s'_{-i})$$

By equation 48, I have the following equations.

$$E(w_i|d_i = \bar{s}, \zeta_i = \underline{s}+1, s_{-i}) =$$
$$\int_{\bar{s}}^{\underline{s}+1} \frac{\underline{s}+1-x}{2}\bar{s}(\underline{s}+1-x)\,dx + (1-\bar{s}+\frac{(\bar{s}-\underline{s})^2}{2})(\underline{s}+1-\bar{s})\frac{\underline{s}+1-\bar{s}}{2}$$

$$E(w_i|d_i = \bar{s}+\varepsilon, \zeta_i = s'+1, s'_{-i}) =$$
$$\int_{\bar{s}+\varepsilon}^{\underline{s}+\varepsilon+1} \frac{\underline{s}+\varepsilon+1-x}{2}\bar{s}(\underline{s}+\varepsilon+1-x)\,dx + (1-\bar{s}+\frac{(\bar{s}-\underline{s}-\varepsilon)^2}{2})(\underline{s}+1-\bar{s})\frac{\underline{s}+1-\bar{s}}{2}$$

$$E(w_i|d_i = \bar{s}, \zeta_i = \underline{s}+1, s_{-i}) > E(w_i|d_i = \bar{s}+\varepsilon, \zeta_i = s'+1, s'_{-i})$$

Lemma 28 shows that $\bar{m}_i > \frac{-(\bar{s}-\underline{s})^3+3(\bar{s}-\underline{s})^2-3\bar{s}+6}{6\bar{s}-3(\bar{s}-\underline{s})^2}$ is implied by $\bar{i}(\underline{s},\bar{s})$. By proposition 4, under assumption 3 for both players, Nash equilibria where assumption 4 holds both players satisfy $\bar{m}_i = \bar{i}(\underline{s},\bar{s}) \le \bar{w}(\underline{s},\bar{s})$. By lemma 21's (2) and lemma 23, if $\bar{m}_i = \bar{i}(s',\bar{s}) \le \bar{w}(s',\bar{s})$, when player $-i$ plays s'_{-i}, player i weakly prefers $d_i = \bar{s}$ with $\zeta_i = s'+1$ to $d_i = \bar{s}+\varepsilon$ with $\zeta_i = s'+1$.

Therefore, using symmetry, in the set of the Nash equilibria, $E(u_i|d_i = \bar{s}, \zeta_i = \underline{s}+1)$ and $E(u_{-i}|d_{-i} = \bar{s}, \zeta_{-i} = \underline{s}+1)$ are increasing in $\underline{s}$. By lemma 25 and formula 33, $E(u_{-i}|d_{-i} = \bar{s}, \zeta_{-i} = \underline{s}+1) = 0$ must hold. This means $\bar{m}_{-i}$ must be lower for a higher $\underline{s}$. By lemma 25, in the set of the Nash equilibria, $\bar{i}(\underline{s},\bar{s})$ is decreasing in $\underline{s}$.

Lemma 27 states that when $\underline{s} = 0$, $s_p(\bar{s}) > 0$. Therefore, by lemma 29, when $\underline{s} = 0$, $\bar{i}(\underline{s},\bar{s}) > \bar{w}(\underline{s},\bar{s})$.

Next, I will consider $\bar{w}(\underline{s},\bar{s}) = \frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2}$.

$$\frac{\partial(\frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2})}{\partial \underline{s}} = \frac{1-\bar{s}}{(\bar{s}-\underline{s})^2} - \frac{1}{2} \tag{67}$$

$$\frac{\partial(\frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2})}{\partial \underline{s}} \gtreqless 0 \leftrightarrow \frac{1-\bar{s}}{(\bar{s}-\underline{s})^2} - \frac{1}{2} \gtreqless 0 \leftrightarrow \bar{s}-\underline{s} \lesseqgtr \sqrt{2-2\bar{s}} \tag{68}$$

Consider the case where $\bar{s} \ge \sqrt{2-2\bar{s}}$. Lemma 5 establishes that when $\bar{s}-\underline{s} \ge \sqrt{2-2\bar{s}}$, $\bar{i}(\underline{s},\bar{s}) > \bar{w}(\underline{s},\bar{s})$. $\bar{i}(\underline{s},\bar{s})$ is bounded in $\underline{s}$.

$$\lim_{\underline{s}\to\bar{s}}(\frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2}) = \infty$$

When $\bar{s}-\underline{s} < \sqrt{2-2\bar{s}}$, by formula 67, $\frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2}$ is increasing in $\underline{s}$. From earlier I have that $\bar{i}(\underline{s},\bar{s})$ is bounded and that it is decreasing in $\underline{s}$ in the set of the Nash equilibria. Proposition 3 establishes that under assumption 3, assumption 4 for both players is a Nash equilibrium if and only if $\bar{m}_1 = \bar{m}_2 = \bar{i}(\underline{s},\bar{s}) \le \bar{w}(\underline{s},\bar{s})$. By the intermediate value theorem, there exists some $\underline{s} \in (\bar{s}-\sqrt{2-2\bar{s}}, \bar{s})$ for which $\bar{i}(\underline{s},\bar{s}) = \bar{w}(\underline{s},\bar{s})$. Given the monotonicity of $\bar{i}(\underline{s},\bar{s})$ and $\bar{w}(\underline{s},\bar{s})$ in the set of the Nash equilibria, this intersecting $\underline{s}$ is unique. This is the $\acute{s}$.

Consider the case where $\bar{s} < \sqrt{2-2\bar{s}}$. This means $\bar{s}^2 < 2-2\bar{s}$ and $\bar{s} < \sqrt{3}-1$. By equation 67, $\frac{\partial(\frac{1-\bar{s}}{\bar{s}-\underline{s}} + \frac{\bar{s}-\underline{s}}{2})}{\partial \underline{s}} > 0$.

Earlier in this proof, I showed that when $\underline{s} = 0$, $\bar{i}(\underline{s},\bar{s}) > \bar{w}(\underline{s},\bar{s})$. Therefore, by a logic similiar to before, I have the same result for the $\acute{s}$ as the $\bar{s} \ge \sqrt{2-2\bar{s}}$ case.

Finally, proposition 4.4 proves proposition 5.4. ■

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