DISCUSSION PAPERS IN ECONOMICS

Working Paper No. 14-01

Why is Pigou Sometimes Wrong? Understanding How Distortionary Taxation can Cause Public Spending to Exceed the Efficient Level

Charles A.M. de Bartolome
University of Colorado at Boulder

March 2014

Department of Economics

University of Colorado at Boulder
Boulder, Colorado 80309

© March 2014, Charles A.M. de Bartolome

WHY IS PIGOU SOMETIMES WRONG? UNDERSTANDING HOW DISTORTIONARY TAXATION CAN CAUSE PUBLIC SPENDING TO EXCEED THE EFFICIENT LEVEL

by

Charles A.M. de Bartolome
University of Colorado at Boulder

ABSTRACT

When a public good is financed by a proportional tax, the price distortion increases the marginal cost of the public good above its resource cost. Pigou (1928) conjectured that the higher cost lowers the second-best public good level below the first-best level. I explain how the price distortion is likely to also raise the marginal benefit of the public good, so that Pigou's reasoning is incomplete. The second-best public good level exceeds the first-best level when the price distortion increases the marginal benefit more than the marginal cost.

Key words: proportional tax, second-best, public service level.

Suggested running title: Why is Pigou sometimes wrong?

JEL Classification: H21, H40, H41, H80

Address for correspondence: Charles A.M. de Bartolome, Department of Economics, University of Colorado, 256 UCB, Boulder, CO 80309-0256, USA. email: debartol@colorado.edu

1. INTRODUCTION

With lump-sum taxation, the first-best public good level is characterized by the "Samuelson Rule": expenditure on the public good should be increased until the marginal benefit equals the marginal resource cost.[1] The use of lump-sum taxation to finance a public good is of course unrealistic, but it is a useful benchmark against which to compare other tax structures. If the lump-sum tax is replaced by a proportional tax, there is a price distortion - the consumer price exceeds the marginal cost of the taxed good - which gives rise to a welfare cost; this welfare cost is henceforth termed the “distortion cost”. Pigou (1928, p. 53) recognized that the distortion cost should be added to the resource cost to obtain the “full” marginal cost of the public good, and conjectured that the increased cost would cause the second-best public good level to fall below the first-best level. Atkinson and Stern (1974), King (1986) and Wilson (1991a) confirm that the second-best public good level is indeed less than the first-best level when the household's utility function has either Cobb-Douglas or CES form. However, Pigou’s conjecture is not generally correct: de Bartolome (1998), Gaube (2000) and Gronberg and Liu (2001) provide examples in which the second-best public good level exceeds the first-best level.

Pigou's intuition hinges on the way the proportional tax increases the marginal cost of the public good. What is missing in his analysis - and what is discussed in this paper - is the effect of the distortion on the marginal benefit of the public good.[2] The marginal benefit of the public good is measured as the quantity of numeraire which households are willing to give up to obtain an additional unit of the public good. By changing the bundle of commodities consumed, the distortion changes the household’s willingness to give up the numeraire. In particular, using a proportional tax instead of a lump-sum tax causes the household to substitute into - or increase its consumption of - the untaxed numeraire. This increased consumption is likely to lower the

marginal utility of the numeraire. With the numeraire giving less marginal utility, the household is more willing to give it up to gain the public service, or the marginal benefit of the public good increases. If the marginal benefit increases more than the marginal cost, the net effect of the distortion is to increase the second-best public good level above the first-best level.

That the price distortion affects the marginal benefit of the public good as well as its marginal cost enables me to provide some intuition for the result of Gaube (2000) that, if the second-best public good level exceeds the first-best level, the numeraire must be a Marshallian complement with the taxed good. It also enables me to show that, if the second-best public good level exceeds the first-best level, the taxed commodity must be normal. And it provides intuition for Wilson's (1991b) result. In Wilson's model, there are dissimilar households so that the planner wants to redistribute resources between households in addition to providing the public good. Wilson shows that the second-best public good level may exceed the first-best level and attributes this to the gain of shifting resources from the distorted private sector to the public sector. This paper provides an alternative explanation: redistribution is achieved by using a uniform lump-sum tax financed by a proportional commodity tax. When choosing the public good level, it is "as if" the commodity tax is creating a pre-existing price distortion and, as discussed above, the effect of this price distortion is to raise the marginal benefit of the public good. The presence of a pre-existing price distortion suggests another interesting case: financing the public good with a lump-sum tax when one of the private goods provides a negative externality which is not internalized by a Pigou Tax.

The paper is organized as follows. Section 2 characterizes the first-best and second-best allocations. Section 3 discusses how the distortion associated with the second-best allocation effects the public good level. Section 4 concludes.

2. THE MODELS

The population is comprised of a large number of households, each of which has identical tastes and income. A representative household consumes leisure ℓ, a commodity x, and the public service[3] z. The utility achieved by the household is assumed to be additively separable between private goods and the public service, or is

$$U(\ell, x) + G(z).$$

$U(.,.)$ and $G(.)$ are smooth, strictly concave functions. This utility structure is often used in the literature; it has the advantage that commodity demands are independent of the public service.[4]

Leisure is the numeraire. The household has a time endowment H (units of leisure). The resource cost of producing one unit of commodity x is p units of numeraire and, with perfect competition, this is the producer price; the resource cost of producing one unit of the public service is k units of numeraire. The resource constraint of the economy is

$$\ell + px + kz = H. \tag{1}$$

The allocation problem is envisioned as a two-stage Stackelberg game. At the first stage the government chooses the public service level and the tax structure, and at the second stage each household chooses its consumption ℓ and x. The government's choice of the public service level and the tax rates at the first stage is constrained: its budget must balance if households make their predicted choices at the second stage.

2.1 First-best Allocation.

The benchmark analysis is the first-best case in which the public service is financed by a lump-sum tax imposed at the first-stage. At the second stage, the household takes the tax kz as given. Equation (1) is the household's budget constraint and, using Equation (1) to substitute for ℓ, the household chooses x as

$$\max_{x} \quad U(H - px - kz, x) + G(z) \qquad \text{s.t. } z \text{ given.}$$

The choice $x^F(z)$ is defined implicitly by the the first-order condition

$$\frac{U_x(H - px^F(z) - kz, x^F(z))}{U_\ell(H - px^F(z) - kz, x^F(z))} = p \ , \tag{2}$$

where the subscript identifies the position of differentiation.[5] Each household chooses x such that its marginal benefit equals its resource cost. Using Equation (1), each household's consumption of the numeraire is $\ell^F(z) \equiv H - px^F(z) - kz$.

In the first stage of the maximization, the government chooses z knowing the household's response $x^F(z)$ at the second stage.

$$\max_{z} \quad U(H - px^F(z) - kz, x^F(z)) + G(z) \ .$$

The first-order condition is obtained by differentiating with respect to z, using Equation (2) and rearranging

$$\frac{G_z(z)}{U_\ell(H - px^F(z) - kz, x^F(z))} = k \ . \tag{3}$$

The left-hand side of Equation (3) is the marginal benefit of the public service when financed by a lump-sum tax and is denoted as $MB^F(z)$; the marginal benefit is measured as the amount of numeraire the household is willing to give up to gain an extra unit of z. The right-hand side of Equation (3) is the marginal cost of the public service when financed by a lump-sum tax and is denoted $MC^F(z)$. It is shown in Appendix A that $MB^F(z)$ is a decreasing function of z.

Figure 1 (shown in the next section) shows both the first-best and second-best allocations. Figure 1(a) considers the allocation of ℓ and x, and Figure 1(b) considers the determination of z. In this section we are considering the first-best allocation. In Figure 1(a), AB is the household's budget line if $z = 0$. If $z > 0$, the lump-sum tax shifts the household's budget line to CD and the household's allocation $(\ell^F(z), x^F(z))$ is at the point of tangency F of the indifference curve I and the budget line CD. In Figure 1(b), $MB^F(z)$ is the downward sloping solid line and $MC^F(z)$ is the horizontal solid line. First-best efficiency requires that the marginal benefit equals the marginal cost, or z^F occurs at the intersection of $MB^F(z)$ and $MC^F(z)$ at G.

2.2 Second-best allocation

The second-best model assumes that the public service must be fully financed by a tax on the commodity x which is imposed at the first-stage. If the tax rate is t, the consumer price of a unit of commodity x is

$$q \equiv p(1+t) \ .$$

At the second stage, each household's budget is $\ell + qx = H$; each household takes q and z as given and chooses x to maximize his utility:

$$\max_x \ U(H - qx, x) + G(z) \quad \text{s.t.} \quad q, z \text{ given.}$$

The household's choice of x conditional on q, $x^S(q)$, is implicitly defined by the first-order condition[6]

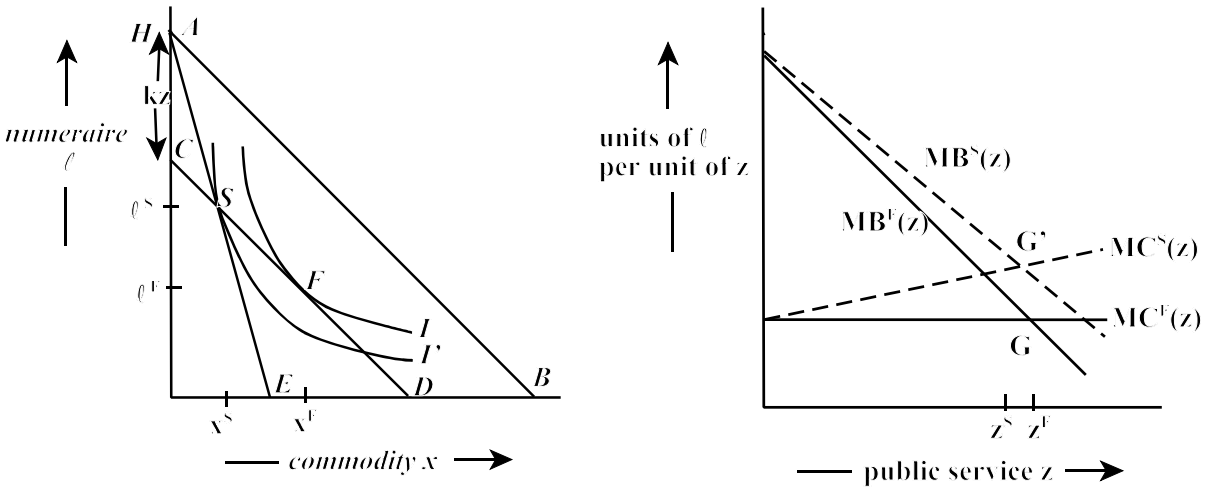
$$\frac{U_x(H - qx^S(q), x^S(q))}{U_\ell(H - qx^S(q), x^S(q))} = q \ . \tag{4}$$

Each household buys x until the marginal benefit equals the consumer price.

At the first stage, the government is restricted to choices of q and z which balance its budget given the household's choice at the second stage, or to choices q and z such that

$$(q - p)x^S(q) = kz. \tag{5}$$

Equation (5) shows that q is a function of z, $q(z)$, and the household choice of x becomes an implicit function of z, $x^S(q(z))$. The household's consumption of the numeraire is $\ell^S(q(z)) \equiv H - qx^S(q(z))$ or, using Equation (5), $\ell^S(q(z)) = H - px^S(q(z)) - kz$.



(a) household choice (b) govt. choice

Figure 1: first- and second-best efficiency with commodity taxation

In Figure 1(a), the commodity tax raises the consumer price of x above p, or the household's budget line swivels from AB to AE. If the same level of public service is provided as in the lump-sum case, the government must choose the tax rate so that the same resources kz are collected from the household or so that the household's budget line AE induces the household to choose an allocation on CD: facing the budget line AE , the household achieves the indifference curve I' at S. $x^S(q(z))$ is the consumption of x at S. The commodity tax causes a consumption distortion - the household consumes less commodity x and more numeraire than if a lump-sum tax is used. This is formalized in the Lemma below.

LEMMA 1: $\ell^S(q(z)) > \ell^F(z)$.

PROOF:
$$U_x(H - px^F(z) - kz, x^F(z)) - pU_\ell(H - px^F(z) - kz, x^F(z)) = 0$$
$$< \frac{kz}{x^S(q(z))} U_\ell(H - qx^S(q(z)), x^S(q(z))) = (q - p)U_\ell(H - qx^S(q(z)), x^S(q(z)))$$
$$= U_x(H - px^S(q(z)) - kz, x^S(q(z))) - pU_\ell(H - px^S(q(z)) - kz, x^S(q(z))). \quad (6)$$

The first equality is Equation (2), the second inequality follows from the strict concavity of $U(.,.),$, the third equality uses Equation (5) and the fourth Equality uses Equation (4). But

$$\frac{\partial}{\partial x}\left[U_x(H - px - kz, x) - pU_\ell(H - px - kz, x)\right] = p^2 U_{\ell\ell} - 2pU_{\ell x} + U_{xx} < 0, \quad (7)$$

where the last inequality follows from the strict concavity of $U(.,.)$. Inequality (6) and Inequality (7) together imply $x^S(q(z)) < x^F(z)$. Using Equation (5) to substitute for q:

$$\ell^S(q(z)) \equiv H - qx^S(q(z)) = H - px^S(q(z)) - kz > H - px^F(z) - kz \equiv \ell^F(z). \ \square$$

At the first stage, the government chooses the public service level (and the implied tax rate) as

$$\max_{z} \; U(H - q(z)\, x^{S}(q(z)),\; x^{S}(q(z))) \;+\; G(z).$$

Differentiating to obtain the first-order condition,

$$U_{\ell}\left(- q\frac{dx^{S}}{dq}\frac{dq}{dz} - x^{S}\frac{dq}{dz}\right) \;+\; U_{x}\frac{dx^{S}}{dq}\frac{dq}{dz} \;+\; G_{z} \;=\; 0 \;.$$

Using Equation (4), the first-order condition is written as

$$\frac{G_{z}(z)}{U_{\ell}(H - q(z)x^{S}(q(z)),\, x^{S}(q(z)))} \;=\; x^{S}\frac{dq}{dz} \;. \tag{8}$$

Differentiating Equation (5) and rearranging,

$$x^{S}\frac{dq}{dz} \;=\; k\,\frac{1}{1 - \dfrac{q - p}{q}\,\varepsilon} \;,$$

where $\varepsilon \equiv -\,(q/x)\, dx^{S}/dq$ is the price elasticity of demand for commodity x.

ASSUMPTION 1: the price elasticity of the taxed commodity is positive, or $\partial x/\partial q < 0$.

Substituting for $x^{S}\, dq/dz$ in Equation (8), the second-best public service z^{S} is the solution to

$$\frac{G_{z}(z)}{U_{\ell}(H - q(z)x^{S}(q(z)),\, x^{S}(q(z)))} \;=\; k\,\frac{1}{1 - \dfrac{q(z) - p}{q(z)}\,\varepsilon} \;. \tag{9}$$

The left-hand side of Equation (9) is denoted $MB^S(z)$, the marginal benefit of the public service when financed by the commodity tax. The right-hand side, being the marginal resource cost multiplied by the marginal cost of funds[7] , is the marginal cost of the public service when financed by the commodity tax, denoted $MC^S(z)$. The assumption that $\varepsilon > 0$ implies that the right-hand side of the Equation (9) exceeds k or that, in Figure 1(b), $MC^S(z)$ lies strictly above $MC^F(z)$ for $z > 0$.

The second-best efficient level z^S occurs at the intersection G' of $MB^S(z)$ and $MC^S(z)$. Whether $z^F > z^S$ (as Pigou conjectured) or $z^S > z^F$ depends on whether and by how much $MB^S(z)$ lies above $MB^F(z)$. This is the topic to which I now turn.

3. DISTORTION AND PUBLIC SERVICE LEVEL

3.1 The shift in the marginal benefit curve

The focus of this paper is how the distortion associated with the use of the commodity tax shifts the marginal benefit schedule of the public service. Lemma 1 establishes that, at any value of z, using the proportional tax causes the household to consume more numeraire and less of the taxed good than when a lump-sum tax is used, or to move along CD from F to S in Figure 1. This affects the marginal utility U_ℓ. Each small movement along CD away from F and towards S affects the marginal benefit schedule at a pre-determined z as

$$\frac{d}{d\ell}\,\frac{G_z(z)}{U_\ell\left(\ell\,,\,\dfrac{H-\ell-kz}{p}\right)} \;=\; -\;\frac{G_z}{U_\ell^2}\left[U_{\ell\ell}-\frac{U_{\ell x}}{p}\right]. \tag{10}$$

The first term in the bracket is the effect of the increase in leisure on U_ℓ. Because $U_{\ell\ell} < 0$, the increase in leisure lowers the marginal utility of leisure and thereby increases household's willingness to give up leisure to get the public service. The second term is the effect of the decrease in x on U_ℓ. If $U_{\ell x} > 0$, the decrease in the quantity of x also lowers the marginal utility of leisure, reinforcing the first effect and further pushing up the marginal benefit schedule. Overall, provided $U_{\ell x} > pU_{\ell\ell}$ at all points along FS, the distortion increases the marginal benefit of the public service at public service level z.

3.2 Normality/Inferiority

The total shift in the marginal benefit schedule as the household's allocation moves from F to S is the sum of the incremental changes, or is

$$MB^F(z) - MB^S(z) = \int_{\ell^F(z)}^{\ell^S(q(z))} \frac{G_z}{U_\ell^2}\frac{1}{p}\left[-pU_{\ell\ell} + U_{\ell x}\right]d\ell \ . \qquad (11)$$

Denoting the slope of the indifference curve between ℓ and x at a point as Q (i.e. $Q \equiv U_x / U_\ell$), x is normal at that point if[8]

$$-QU_{\ell\ell} + U_{\ell x} > 0.$$

At F, $Q = p$ so that normality implies that the "first" incremental changes from F to S shift the marginal benefit curve up. As the allocation moves further along CD towards S, Q increases. With $Q > p$, $-QU_{\ell\ell} + U_{\ell x} > 0$ does not ensure that $-pU_{\ell\ell} + U > 0$; in this case the incremental changes in ℓ close to S could lead to incremental decreases in the marginal benefit; put differently x being normal is insufficient to ensure that the overall effect is an increase in the marginal benefit.

If x is inferior, $-Q\,U_{\ell\ell} + U_{\ell x} < 0$. With $Q \geq p$, this is sufficient to ensure that $-p\,U_{\ell\ell} + U_{\ell x} < 0$ at all points on FS, and the marginal benefit schedule shifts down. In this case, $MB^S(z)$ lies below $MB^F(z)$

Summarizing, x being normal favors but does not ensure that moving to the second-best from the first-best shifts up the marginal benefit schedule. However, x being inferior does ensure that the marginal benefit schedule shifts down. The link to the price distortion is that large $U_{\ell x}$ is favorable to both x being normal and to the marginal benefit curve shifting up.

3.3 Comparison of z^F and z^S

The second-best public service level, z^S, lies at the intersection of $MB^S(z)$ and $MC^S(z)$ in Figure 1(b). $\varepsilon > 0$ implies $MC^S(z) > MC^F(z)$. If $MB^S(z) > MB^F(z)$, both curves shift up and it is potentially ambiguous whether the second-best public service level is less or greater than the first-best level. Appendix B provides an example in which the shift in the marginal benefit schedule exceeds the shift in the marginal cost schedule or in which $z^S > z^F$.

As noted above, if x is inferior at all points on FS, then $MB^S(z) < MB^F(z)$; with $\varepsilon > 0$, $MC^S(z) > MC^F(z)$ and hence $z^S < z^F$. This in turn implies Lemma 2:

LEMMA 2: If $\varepsilon > 0$ and $z^S > z^F$, then at some points on FS the taxed commodity is normal.

Gaube (2000) assumes that x is normal and the possibility of x being inferior is not explored.

Henceforth I assume that x is normal. In fact I make the slightly stronger assumption as

ASSUMPTION 2: At all points on the line joining the first-best and the second-best allocations in (ℓ, x) space, $-p\,U_{\ell\ell} + U_{\ell x} > 0$.

Assumption 2 ensures that the distortion shifts up the marginal benefit schedule of the public service.

3.4 Substitutability/Complementarity

Several results in the literature depend on whether ℓ and x are substitutes or complements. E.g. Gaube (2000, Proposition 1) shows that, if ℓ and x are normal and if ℓ is a Marshallian substitute for x (i.e. $\partial l / \partial q > 0$), then $z^F > z^S$. Put differently, if ℓ and x are normal goods, a necessary (but not sufficient) condition for $z^S > z^F$ is that ℓ is a Marshallian complement for x. The importance of complementarity can be understood from Equation (10). Loosely, ℓ and x are complements when ℓ gives more utility if it is used in the presence of x, or complementarity is favored if $U_{\ell x} > 0$. [9] As noted in Section 3.1, $U_{\ell x} > 0$ implies that, if a commodity tax replaces the lump-sum tax, the implied decrease in x lowers the marginal utility of ℓ, increasing the upward shift in $MB^S(z)$. This favors z^S exceeding z^F.

3.5 Wilson's (1991b) model

Wilson (1991b) provides an example in which the second-best public service exceeds the first-best. His model has many dissimilar households, and the government must determine the extent of redistribution and the public service level. In the first-best, redistribution and the financing of the public service are achieved using individualized lump-sum taxes. In the second-best, a commodity tax is used for redistributional concerns and any additional tax revenue is collected using a uniform lump-sum tax. Although the commodity tax, the lump-sum tax and the public service level are determined simultaneously, the second-best analysis is done with the commodity tax being pre-determined (at its optimal level) so that any marginal change in the

public service level is financed by changing the lump-sum tax. In this framework Wilson provides an example in which the second -best public service level exceeds the first-best level.

Using the framework of a representative household (in which there can be no redistributional motives), we can see the forces at work by supposing that in the first-best there is only a lump-sum tax but that in the second-best there is a pre-existing commodity tax rate $\bar{q}$ (potentially financing a lump-sum transfer) and that any shortfall in the tax-receipts required to finance the public service is made-up by a lump-sum tax. The first-best analsysis has already been described in Section 2.1 and I now turn to the second-best analysis. At the second stage of the Stackelberg game, the household faces the consumer price $\bar{q}$ for *x* and a lump-sum tax *T*, or the household's problem is:

$$\max_{\ell, x} U(\ell, x) \quad \text{s.t.} \quad \ell + \bar{q}x = H - T .$$

The solutions are: $\ell(\bar{q}; H - T)$ and $x(\bar{q}; H - T)$. At the first stage, the government's problem is:

$$\max_{z, T} U(\ell(\bar{q}; H - T), x(\bar{q}; H - T)) \quad \text{s.t.} \quad (\bar{q} - p)\, x(\bar{q}; H - T) + T = kz .$$

Using the household budget constraint to substitute for ℓ and recognizing that the government's budget constraint implies that *T* is a function of *z*, at the first stage the government's problem is

$$\max_{z} U(H - T(z) - \bar{q}x(\bar{q}; H - T(z)), x(\bar{q}; H - T(z))) + G(z).$$

Using *M* to denote the income position, differentiate to find the optimal:

$$\frac{\partial U}{\partial \ell}\left(- \frac{dT}{dz} + \bar{q}\frac{\partial x}{\partial M}\frac{dT}{dz}\right) - \frac{\partial U}{\partial x}\frac{\partial x}{\partial M}\frac{dT}{dz} + G_z(z) = 0.$$

But the first-order condition for the household problem gives $U_\ell - (1/\bar{q})U_x = 0$; hence

$$\frac{G_z(z)}{U_\ell(\ell(\bar{q};H-T(z),\, x(\bar{q};H-T(z)))} = \frac{dT}{dz}\,. \tag{12}$$

To find *dT/dz*, differentiate the government budget constraint:

$$-(\bar{q}-p)\frac{\partial x}{\partial M}\frac{dT}{dz} + \frac{dT}{dz} = k$$

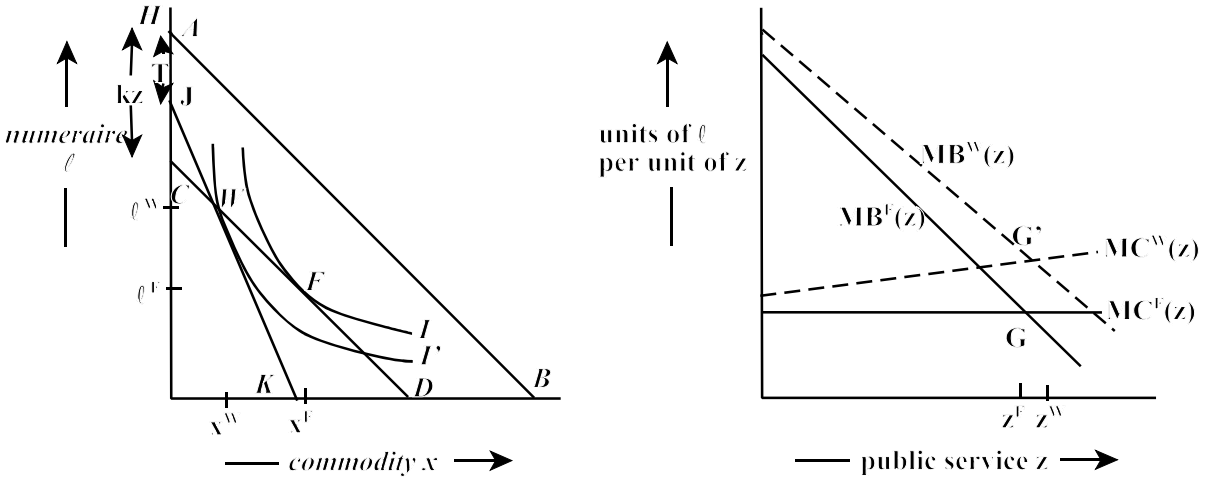
or

$$\frac{dT}{dz} = \frac{k}{1-(\bar{q}-p)\dfrac{\partial x}{\partial M}}\,.$$

Substituting for *dT/dz* in Equation (12),

$$\frac{G_z(z)}{U_\ell(\ell(\bar{q};H-T(z),\, x(\bar{q};H-T(z)))} = k\,\frac{1}{1-(\bar{q}-p)\dfrac{\partial x}{\partial M}}\,. \tag{13}$$

The left-hand side of Equation (13) is the second-best marginal benefit and is denoted $MB^W(z)$. The right-hand side is the second-best marginal cost, being the resource cost multiplied by the marginal cost of funds, and is denoted $MC^W(z)$.



(a) household choice (b) govt. choice

Figure 2: first- and second-best efficiency for the Wilson model

In Figure 2(a), *AB* is the economy's budget line if *z=0* (and the price of *x* is its marginal resource cost *p*). If the public service is *z*, then *kz* must be withdrawn from the household or the budget line of the economy is *CD*. *F* is the first-best allocation (conditional on *z*). If the same level of the public service is provided but the consumer price of *x* is now $\bar{q}$, $\bar{q} > p$, and the lump-sum tax is *T* , $T < kz$, the consumer's budget line is *JK*. The consumer's choice *W* is where his indifference curve just touches his budget line and the government must chose the lump-sum tax *T* so that *W* lies on *CD*. Comparing *W* with *F*, the price distortion $\bar{q}$ causes the chosen bundle to contain more ℓ and less *x* than in the first-best.

Moving to Figure 2(b) and using an argument analogous to that used in Section 3.1, Assumption 2 implies that the increase in the consumption of ℓ associated with moving from *F* to *W* raises $MB^W(z)$ above $MB^F(z)$. (In this case $\bar{q} > p$ and hence, even at $z = 0$, $MB^W(0) > MB^F(0)$). $MC^W(z)$ also lies strictly above $MC^F(z)$.[10] In Wilson's example $z^S > z^F$ arises because the pre-existing tax distortion (caused by the desire for redistribution) increases the marginal benefit of the public service by more than it increases the marginal cost (relative to the first-best).

3.6 Extension: unpriced externality

In the previous examples the second-best arises because *x*'s consumer price *q* lies above it's social price *p*. In this subsection we extend the analysis to consider the case when the consumption of *x* is associated with an negative externality; in the second-best analysis there is no Pigou tax, so that *x*'s consumer price lies below it's social price, and the public service is financed by a lump-sum tax. Intuitively, the household consumes "too much" *x* and "too little" ℓ

(relative to the first-best): the logic of the previous sections suggests that this bundle distortion typically lowers the marginal benefit schedule, ceteris paribus favoring $z^S < z^F$.

Maintaining the structure of the previous sections, utility is considered to have the form:

$$U(\ell, x) - A(E) + G(z)$$

where E represents an aggregate negative externality associated with the consumption of x. If N is the number of households, $E = Nx$. In addition, $A_E > 0$. Using the economy's resource constraint $\ell + px + kz = H$ to substitute for ℓ, the first-best planner's problem is:

$$\max_z \max_x \; U(H - px - kz, x) - A(Nx) + G(z).$$

As in Section 2, write the solution to the second-stage as $x^f(z)$. It is defined by the first-order condition:

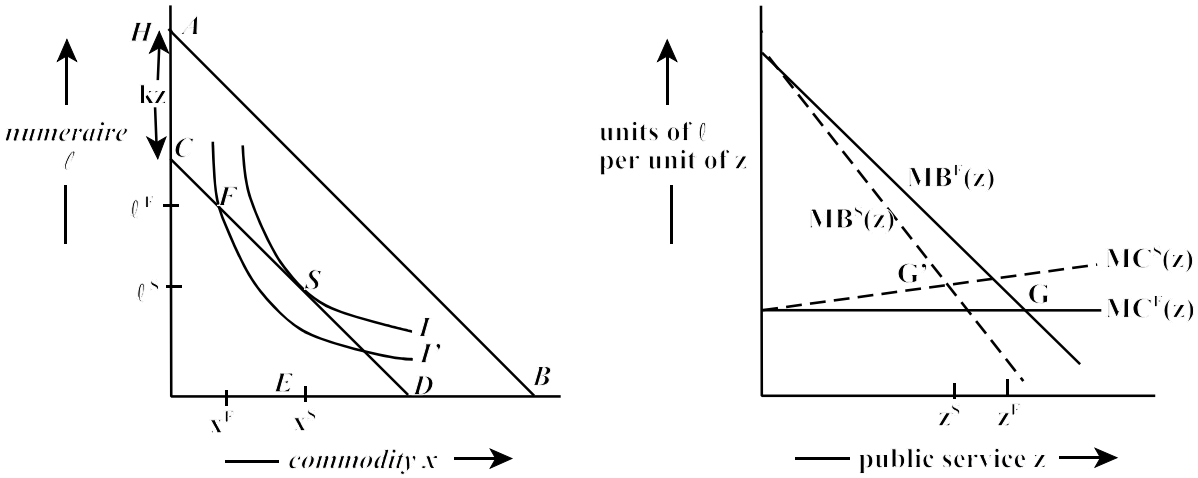
$$\frac{U_x}{U_\ell} = p + N\frac{A_E}{U_\ell} > p.$$

The slope of the household's indifference between ℓ and x is x's resource cost plus the external cost of consuming a marginal unit of x. At the first stage, the government chooses z such that

$$\frac{G_z(z)}{U_\ell(H - px^f(z) - kz, x^f(z))} = k. \tag{14}$$

The left-hand side of Equation (14) is the marginal benefit of the public service, $MB^f(z)$. The right-hand side is the marginal cost, $MC^f(z)$. In Figure 3(a), AB is the economy's resource constraint if $z = 0$. With a lump-sum tax kz, the resource constraint is CD. The negative

externality raises the social price of x, so the first-best outcome (conditional on z) is the point F on CD where the slope of the indifference curve is $p + N A_E/U_\ell$.



(a) household choice (b) govt. choice

Figure 3. first- and second-best efficiency with externalities

In the second-best problem, the externality is assumed to be unpriced and the public service is financed by a lump-sum tax. At the second stage, the household budget constraint is $\ell = H - px - kz$, the household takes the level of the externality as given and the household problem is:

$$\max_{x} \quad U(H - px - kz, x) - A(E) + G(z) \quad \text{s.t.} \quad z, E \text{ given.}$$

Because of additive separability, the solution does not depend on either E or z, and is $x(p; H - kz)$; $x(p; H - kz)$ is defined by the first-order condition

$$\frac{U_x(H - px(p; H - kz) - kz, x(p; H - kz))}{U_\ell(H - px(p; H - kz) - kz, x(p; H - kz))} = p \tag{15}$$

At the first stage the government problem is:

$$\max_{z} \; U(H - px(p;H-kz) - kz, x(p;H-kz)) \; - \; A(Nx(p;H-kz)) \; + \; G(z).$$

Using Equation (15), the first order condition is

$$\frac{G_z(z)}{U_\ell} \; + \; \frac{A_E}{U_\ell} N k \frac{\partial x}{dM} \; = \; k \tag{16}$$

where *M* denotes the income position of differentiation. The left-hand side of Equation (16) is the marginal benefit of the public service and it contains two terms. The first term is the marginal benefit of the public service *per se*. This term is affected by the distortion. In particular, ℓ is decreasing as the allocation moves from *F* to *S* in Figure 3(a); using Equation (10), incremental decreases in ℓ decrease the marginal benefit if $-p\,U_{\ell\ell} + U_{\ell x} > 0$ which is the case assumed in Assumption 2, or the effect of the distortion is to shift the marginal benefit schedule down. The second term is the external benefit of the reduced consumption of *x* induced by the increased lump-sum tax; this benefit is not present in the first-best and the addition of this benefit shifts the marginal benefit schedule up. The right-hand side of Equation (16) is the marginal cost. I leave to future research whether the overall effect is to raise or lower the public service: the focus of this paper is the effect of the distortion on the first term and the consequent effect on the marginal benefit schedule.

4. CONCLUSION

This paper explains how the distortion associated with a proportional tax affects the marginal benefit as well as the marginal cost of providing a public service. In doing so, it seeks to integrate the discussion about "rules" and "levels" in the earlier literature and to explain existing examples for which the second-best public service level exceeds the first-best level.

APPENDIX A : DIMINISHING MARGINAL BENEFIT

To show that the first-best marginal benefit curve is downward sloping, differentiate the marginal benefit with respect to z

$$\frac{d}{dz}\ \frac{G_z(z)}{U_\ell(H - px^F(z) - kz,\ x^F(z))}\ =\ \frac{1}{U_\ell^2}\left(U_\ell G_{zz} - G_z\left(\ U_{\ell\ell}(\ -p\frac{dx^F}{dz} - k)\ +\ U_{\ell x}\frac{dx^F}{dz}\right)\right) \quad .(A.1)$$

To obtain dx^F/dz, differentiate Equation (2) with respect to z, and rearrange

$$\frac{dx^F}{dz}\ =\ k\frac{-pU_{\ell\ell}\ +\ U_{\ell x}}{p^2U_{\ell\ell} - 2pU_{\ell x} + U_{xx}}\ .$$

Substitute into Equation (A.1),

$$\frac{d}{dz}\ \frac{G_z(z)}{U_\ell(H - px^F(z) - kz, x^F(z))}\ =\ \frac{1}{U_\ell^2}\left(\ U_\ell G_{zz} + kG_z\frac{(U_{\ell\ell}U_{xx} - U_{\ell x}^2)}{p^2U_{\ell\ell} - 2pU_{\ell x} + U_{xx}}\right)\ .$$

Strict concavity of $G(.)$ implies $G_{zz} < 0$; strict concavity of $U(.,.)$ implies $p^2U_{\ell\ell} - 2pU_{\ell x} + U_{xx} < 0$ and $U_{\ell\ell}U_{xx} - U_{\ell x}^2 > 0$. Hence each term on the right-hand side is negative.

APPENDIX B: AN EXAMPLE WHEN $z^S > z^F$

I now provide an example in which $z^S > z^F$. This example is based on de Bartolome (1998). The demand for commodity x is iso-elastic as[11]

$$x = \frac{C M^b}{q^a}, \quad \text{with } 0 \le a, \ 0 \le C, \tag{B.1}$$

and M is endowed income. With no transfers, $M = H$. The budget constraint gives the demand for the numeraire

$$\ell = M - qx = M - CM^b q^{1-a}. \tag{B.2}$$

The indirect utility function which generates these demand functions is recovered in Appendix C and is:

$$V(q, M) = M^{1-b} - C\frac{1-b}{1-a} q^{1-a} .$$

We restrict attention to the intervals $a \in [0,1]$ and $b \in [0,1]$. The acceptable set $\{a , b\}$ is further restricted because (1) $\ell \ge 0$,$x \ge 0$ and (2) the implied utility function needs to be concave. With a and b satisfying these conditions, Appendix D shows that further analysis shows that : (i) $U_{\ell x} \ge 0$; (ii) x is a normal good ; (iii) ℓ is a Marshallian complement, $\partial \ell / \partial q < 0$.

For ease of calculation, I set

$$G(z) \equiv \sqrt{z}.$$

With no transfers, $M = H$ and the first-best public service level is:

$$z^F = \underset{z}{argmax} \ (H - kz)^{1-b} - C\frac{1-b}{1-a} p^{1-a} + \sqrt{z} \ ;$$

The second-best public service level is, with abuse of notation,

$$z^S = \underset{q,z}{argmax}\ H^{1-b} - C\frac{1-b}{1-a}\,q^{1-a} + \sqrt{z} \qquad \text{s.t.} \qquad (q-p)\frac{CM^b}{q^a} = kz\ .$$

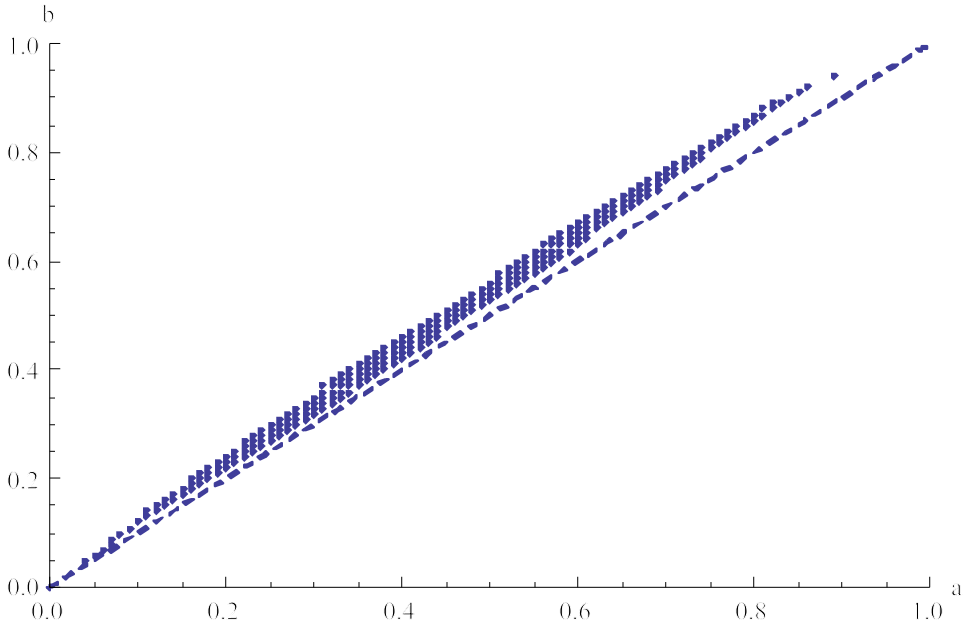


Figure 4: some values of price and wealth elasticities for which $z^S > z^F$

I choose parameter values $p = 1$, $C = ½$, $k = 1$ and $H = 1$, and simulate using the computer. In Figure 4, the dashed line corresponds to the line $b = a$. The area above the shaded area is not permitted because, at the optimum characterized by Equations (3) and (9), the implied utility function $U(.,.)$ is not concave. The shaded area represents permitted values of (a,b) for which $z^S > z^F$; the area below the shaded area represents permitted values of (a,b) for which $z^F > z^S$.

APPENDIX C: RECOVERING THE INDIRECT UTILITY FUNCTION

This procedure follows Hausman (1981). Denote the expenditure required to achieve utility u at consumer price Q as $e(Q, u)$ and denote the compensated demand as $h(Q,u)$. Duality implies that $h(Q, u) = x(Q, e(Q, u))$ or $h(Q, u) = C\, e(Q, u)^b / Q^a$. Shepherd's Lemma implies

$$\frac{\partial e(Q,u)}{\partial Q} = \frac{C\, e(Q,u)^b}{Q^a} .$$

Assume $b \neq 1$ and integrate from some reference price P,

$$\int_{e(P,u)}^{e(q,u)} \frac{de}{e^b} = C \int_P^q \frac{dQ}{Q^a} ,$$

or $$\frac{e(q,u)^{1-b} - e(P,u)^{1-b}}{1-b} = C\, \frac{q^{1-a} - P^{1-a}}{1-a} .$$

Set $u = v(q, M)$, the indirect utility achieved at consumer prices q and endowment M. By duality, $M = e(q, v(q, M))$. Rearranging,

$$e(P, v(q,M))^{1-b} - C\, \frac{1-b}{1-a} P^{1-a} = M^{1-b} - C\, \frac{1-b}{1-a} q^{1-a} .$$

The left-hand side is a monotonic transformation of the indirect utility function. Because tastes are unchanged by a monotonic transformation of the utility function, preferences may also be represented by the indirect utility function

$$V(q,M) \equiv e(P, v(q,M))^{1-b} - C\, \frac{1-b}{1-a} P^{1-a} = M^{1-b} - C \frac{1-b}{1-a} q^{1-a} . \qquad \text{(C.1)}$$

APPENDIX D: CONCAVITY, $U_{\ell x} > 0$, NORMALITY, MARSHALLIAN COMPLEMENTARITY

Concavity

Equations (B.1) and (B.2) give *(q,M)* as implicit functions of *(ℓ,x)*: write the functional dependance as *q(ℓ,x)* and *M(ℓ,x)*. Duality implies

$$U(\ell, x) \equiv V(q(\ell, x), M(\ell, x)).$$

Hence

$$\frac{\partial U}{\partial \ell} = \frac{\partial V}{\partial q}\frac{\partial q}{\partial \ell} + \frac{\partial V}{\partial M}\frac{\partial M}{\partial \ell} \quad \text{and} \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial V}{\partial M}\frac{\partial M}{\partial x}. \tag{D.1}$$

We obtain $\partial V/\partial q$ and $\partial V/\partial M$ by differentiating Equation (C.1)

$$\frac{\partial V}{\partial q} = -\frac{C(1-b)}{q^a} \quad \text{and} \quad \frac{\partial V}{\partial M} = \frac{1-b}{M^b}. \tag{D.2}$$

In turn, we obtain $\partial q/\partial \ell$, $\partial M/\partial \ell$, $\partial q/\partial x$ and $\partial M/\partial x$ by differentiating Equations (B.1) and (B.2) with respect to ℓ and x and then solving the four resulting equations:

$$\frac{\partial q}{\partial \ell} = \frac{1}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}} \quad \text{and} \quad \frac{\partial M}{\partial \ell} = \frac{\frac{a}{b}\frac{M}{q}}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}}. \tag{D.3}$$

$$\frac{\partial q}{\partial x} = -\frac{\frac{M}{b} - q\frac{CM^b}{q^a}}{\frac{CM^b}{q^a}\left(\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}\right)} \quad \text{and} \quad \frac{\partial M}{\partial x} = -\frac{\frac{(1-a)M}{b}}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}}. \tag{D.4}$$

Using Equations (D.2)-(D.4) to substitute into Equations (D.1) and simplify:

$$\frac{\partial U}{\partial \ell} = -C\frac{1-b}{q^a}\frac{1}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}} + \frac{1-b}{M^b}\frac{\frac{a}{b}\frac{M}{q}}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}} = \frac{1-b}{M^b}; \tag{D.5}$$

To obtain $U_{\ell x}$, differentiate with respect to x and use Equation (D.4)

$$\frac{\partial^2 U}{\partial x \partial \ell} = \frac{\partial}{\partial x}\frac{\partial U}{\partial \ell} = -\frac{(1-b)b}{M^{b+1}}\frac{\partial M}{\partial x} = \frac{1-b}{M^b}\frac{1-a}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}}. \tag{D.6}$$

Expressions for $U_{\ell\ell}$, U_x and U_{xx} are derived analogously. Differentiate Equation (D.5) with respect to ℓ and use Equations (D.3) to obtain the expression for $U_{\ell\ell}$;

$$\frac{\partial^2 U}{\partial \ell^2} = \frac{\partial}{\partial \ell}\frac{\partial U}{\partial \ell} = -\frac{(1-b)b}{M^{b+1}}\frac{\partial M}{\partial \ell} = -\frac{(1-b)b}{M^{b+1}}\frac{\frac{a}{b}\frac{M}{q}}{\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}}. \tag{D.7}$$

Use Equations (D.2) and (D.4) in Equation (D.1) to show $U_x = (1-b)q/M^b$, and then differentiate with respect to x and use Equations (D.4) to give an expression for U_{xx}.

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial U}{\partial x} = \frac{1-b}{M^b}\frac{\partial q}{\partial x} - \frac{(1-b)b}{M^{b+1}}q\frac{\partial M}{\partial x} = \frac{1-b}{M^b}\frac{1}{\frac{CM^b}{q^a}\left(\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a}\right)}\left(-\frac{M}{b} + (2-a)q\frac{CM^b}{q^a}\right). \tag{D.8}$$

Concavity implies $U_{\ell\ell}U_{xx} - U_{\ell x}^2 > 0$. Substituting for $U_{\ell\ell}$, U_{xx} and $U_{\ell x}$ using Equations (D.6)-(D.8), concavity implies

$$\frac{a}{b}\frac{M}{q} - \frac{CM^b}{q^a} > 0.$$

$$U_{\ell x} > 0$$

Using Equation (D.6), $0 \le a \le 1$ and $0 \le b \le 1$ and concavity imply $U_{\ell x} > 0$.

Normality of x

Differentiating Equation (B.1),

$$\frac{\partial x}{\partial M} = \frac{b}{M}\frac{CM^b}{q^a} > 0$$

Marshallian Complementarity of ℓ

From Equation (B.2), $\ell = M - CM^b q^{1-a}$. Differentiating with respect to q,

$$\frac{\partial \ell}{\partial q} = -(1-a)\frac{CM^b}{q^a}$$

Marshallian complementarity requires $\partial \ell / \partial q \le 0$ or $a \le 1$.

REFERENCES

Atkinson, A.B., & Stern,N.H. (1974). Pigou, taxation and public goods. *Review of Economic Studies*, 41, 119-128.

Batina, R.G., & Ihori, T.(2005). *Public Goods: Theories and Evidence*. Berlin: Springer.

Chang, M.C. (2000). Rules and levels in the provision of public goods: the role fo complementarities between the public good and the taxed commodity. *International Tax and Public Finance*, 7, 83-91.

de Bartolome, C.A.M., (1998). Is Pigou wrong? Can distortionary taxation cause public spending to exceed the efficient level? Mimeo.

Diamond, P.A., and & Mirrlees, J.A. (1971). Optimal taxation and public production II: tax rules. *American Economic Review*, 61, 261-278.

Gaube, T. (2000). When do distortionary taxes reduce the optimal supply of public goods? *Journal of Public Economics*, 76, 151-180

Gronberg, Y., & Liu, L. (2001). The second-best level of a public good: an approach based on the marginal excess burden. *Journal of Public Economic Theory*, 3, 431-453.

Hausman, J. (1981). Exact consumer's surplus and deadweight loss, *American Economic Review*, 71, 662-676.

King, M.A.. (1986). A Pivovian rule for the optimum provision of public goods. *Journal of Public Economics*, 30, 273-291.

Ng, Y-K. (2000). The optimal size of public spending and the distortionary cost of taxation. *National Tax Journal*, 53, 253-272.

Pigou, A.C. (1928). *A study in public finance* . London: Macmillan.

Samuelson, P.A. (1954). The pure theory of public expenditure. *Review of Economics and Statistics*, 36, 387-389.

Wilson, J.D. (1991a). Optimal public good provision in the Ramsey tax model. *Economic Letters*, 35, 57-61.

Wilson, J.D. (1991b). Optimal public good provision with limited lump-sum taxation. *American Economic Review* 81, 153-166.

ENDNOTES

1. "Marginal benefit" is used synonymously with "marginal rate of substitution" throughout the paper. Similarly, "first-best" and "second-best" are used synonymously with "first-best efficient" and "second-best efficient."

2. Ng (2000, p. 256) and Batina and Ihori (2005, p. 40) note that the marginal benefit curve is affected by a commodity tax. Chang (2000) is also a useful reference on this topic.

3. The assumption of a public service is made to simplify the presentation. The results apply if the government expenditure is on a public good.

4. If the public service is complementary with the taxed commodity, an increase in the public service interacts with the pre-existing tax structure to create additional tax revenue, lowering the marginal cost of funds (Diamond and Mirrrlees (1971)). Separability avoids this additional effect.

5. The chosen x depends on its price p and household income M, and is written in traditional notation as $x(p; M)$. I set $x^F(z) \equiv x(p; H - kz)$.

6. The chosen x depends on the consumer price q and household income M, and is written in traditional notation as $x(q; M)$. I set $x^S(q) \equiv x(q; H)$.

7. The marginal cost of funds measures the units of numeraire the household needs as compensation if one unit of additional tax revenue is raised using tax instrument q. Writing tax revenue as R, the tax rate as $q(R)$ and the expenditure function as $e(q, U)$,

$$MCF = \frac{\partial e}{\partial q}\frac{\partial q}{\partial R}.$$

Using Shephard's Lemma, $\partial e/\partial q = x$. $R = (q-p)x$; differentiating with respect to R,
$1 = x\,\partial q/\partial R + (q-p)\,\partial x/\partial q\,\partial q/\partial R$ or $\partial q/\partial R = 1/(x + (q-p)\,\partial x/\partial q) = 1/(x\,(1 - ((q-p)/q)\,\varepsilon))$. Hence

$$MCF = \frac{1}{1 - \frac{q-p}{q}\varepsilon}.$$

8. Denoting household income as M, the first-order condition for x to be chosen is
$-Q\,U_\ell(M-Qx, x) + U_x(M-Qx, x) = 0$. Differentiate with respect to M and rearrange

$$\frac{\partial x}{\partial M} = \frac{-QU_{\ell\ell} + U_{\ell x}}{-Q^2U_{\ell\ell} + 2QU_{\ell x} - U_{xx}}.$$

Concavity of U implies that the denominator is positive. Therefore x is normal if $-QU_{\ell\ell}+U_{\ell x}>0$.

9. Ng (2000) also notes the importance of complementarity. The household's first-order condition is: $-q\,U_{\ell}(\ell, (H-\ell)/q) + U_{x}(\ell, (H-\ell)/q) = 0$. Differentiate with respect to q and rearrange

$$\frac{\partial \ell}{\partial q} = \frac{qU_{\ell} - (H-\ell)U_{\ell x} + \frac{H-\ell}{q}U_{xx}}{-q^2U_{\ell\ell} + 2qU_{\ell x} - U_{xx}}.$$

The concavity of U implies that the denominator is positive. Therefore $\partial \ell/\partial q \leq 0$ is favored if $U_{\ell x} \geq 0$.

10. At low z levels, the distortion associated with financing using a proportional tax is small so that $MC^W(z) > MC^S(z) \sim MC^F(z)$. At higher z levels, using a lump-sum tax instead of a proportional tax to finance marginal changes in the public service is likely to lower the marginal cost, $MC^S(z) > MC^W(z) > MC^F(z)$. In particular, with a proportional tax, $MC^S(z) = 1/(1+(q-p)(1/x)\,\partial x/\partial q)$. With the lump-sum tax in the presence of a pre-existing tax, $MC^W(z) = 1/(1-(\bar{q}-p)\,\partial x/\partial M)$. At z such that $q = \bar{q}$, the Slutsky Equation implies that $\partial x/\partial q < -x\,\partial x/\partial M < 0$ or that $1/(1+(q-p)(1/x)\,\partial x/\partial q) > 1/(1-(\bar{q}-p)\,\partial x/\partial M)$.

11. When describing a marginal cost of funds schedule which increases with increasing public expenditure, it is normal to presuppose a constant elasticity ε.