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Oracle Efficiency in Additive Partially Linear Triangular Systems

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Abstract: In this paper I propose a two-stage mixed sieve and kernel estimator for the finite dimensional parameter of a partially linear regression model in a triangular system of equations. The model consists of D+1 equations; a single partially linear primary equation having a mixture of endogenous and exogenous regressors, as well as D fully nonparametric secondary equations with exogenous regressors. Regressor endogeneity in the primary equation is handled using the control function approach of Newey et al. (1999). The estimator realizes efficiency gains by imposing an additive structure on the nonparametric component functions of the primary equation and secondary equations of the system (Yu et al. (2011)). As an added benefit, the additive structure circumvents the curse of dimensionality associated with nonparametric estimators. In particular, I show that the estimator of the parametric component β_1 is consistent, $\sqrt{n}$ asymptotically normally distributed, and Oracle efficient having an asymptotic covariance matrix equal to one derived from an identical estimation procedure for a model consisting solely of the primary equation where all regressors are exogeneous. I propose a consistent and easy to compute estimator for the asymptotic covariance matrix of the estimator for β_1.

Keywords: Partially Linear Regression; Endogeneity; Sieve Estimation; Kernel Estimation; Structural Model; $\sqrt{n}$ Asymptotic Normality; Nonparametric Modeling.

JEL Classifications. C13, C14

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1 Introduction

The treatment of endogeneity in semiparametric models has been the subject of intense interest in recent years. In particular, since Newey et al. (1999) in which the control function approach to endogeneity in nonparametric models is established, triangular systems of equations have garnered a great deal of attention see, inter alia, Geng et al. (2016), Ozabaci et al. (2014), Su and Ullah (2008), and Martins-Filho and Yao (2012). The value of the control function approach is clear, providing a method of dealing with endogeneity that minimizes the risk of model misspecification and is easy to implement. I use a control function to handle endogeneity in a partially linear model. Accordingly, this model consists of D+1 equations; a single partially linear primary equation (1) with a mixture of endogenous and exogenous regressors, as well as D fully nonparametric secondary equations (2) having only exogenous regressors. Thus, consider the following triangular system of equations where a set of exogenous regressors enters the primary equation (1) parametrically while allowing endogenous regressors to enter nonparametrically.

$$Y = \beta_0 + Z'\beta_1 + h(X) + \varepsilon, \tag{1}$$

$$X = m(W) + V, \tag{2}$$

$$E(V|W) = 0, \quad E(\varepsilon|W,X) = E(\varepsilon|V). \tag{3}$$

Y is a scalar random variable, X is a D dimensional vector of endogenous random variables in that $E(\varepsilon|Z,X) = E(\varepsilon|X) \neq 0$. W is a q dimensional random vector, and Z is a $p < q$ dimensional subvector of W. ε and V are random disturbances. $[\beta_0 \ \ \beta_1']' \in \mathbb{R}^{p+1}$ and $h(\cdot)$ are unknown parameters of interest. $m(W) : \mathbb{R}^q \to \mathbb{R}^D$ is a vector of D real valued functions $m_d(W)$ where $d \in \{1, 2, \dots, D\}$.

Specifying a partially linear form for the primary equation has the distinct advantage of allowing one to impose a parametric form for some regressors, when justified, while allowing others to be the arguments of a much broader class of functions. Additionally, Robinson (1988) showed that in models consisting solely of equation (1) and where $E(\varepsilon|Z,X) = 0$, his estimator for $[\beta_0 \ \ \beta_1']'$, the finite dimensional parameter of central interest, is consistent and $\sqrt{n}$ asymptotically normal. Furthermore the asymptotic covariance matrix of the estimator for $[\beta_0 \ \ \beta_1']'$ is equal to the semi parametric efficiency bound while only making mild smoothness assumptions regarding $h(X)$. In terms of model structure this paper and Geng et al. (2016) can rightly be viewed as extensions of the model considered in Robinson (1988) to triangular models as both utilize identification strategies to develop an $\sqrt{n}$ asymptotically normal estimator for the finite dimensional parameter $[\beta_0 \ \ \beta_1']'$ separate, in the sense of not depending on, the estimation of $h(X)$. Although Ozabaci et al. (2014) is primarily concerned with the estimation of a fully nonparametric triangular system, they do shortly discuss a partially linear specification without providing a separate estimator for $[\beta_0 \;\; \beta_1']'$ in the sense described above. Unfortunately a semiparametric lower bound for a model consisting of (1), (2), and (3) has yet to be established, hence no claim to semiparametric efficiency can be made, but the estimator developed in this paper does achieve a kind of Oracle efficiency, discussed below, by imposing an additive structure on $h(X)$ and each component of $m(X)$.

The benefits of imposing an additive structure on nonparametric component functions are two fold. Firstly, Yu et al. (2011) show that there are efficiency gains to be had by imposing additivity on $h(X)$. In this context, Manzan and Zerom (2005) can be viewed as an extension of the Robinson (1988) model by separately identifying and estimating $[\beta_0 \;\; \beta_1']'$, while assuming $h(X)$ to be the sum of univariate nonparametric functions. In fact, like Robinson, Manzan and Zerom (2005) showed that their estimator for $[\beta_0 \;\; \beta_1']'$, is $\sqrt{n}$ asymptotically normal and semiparametrically efficient in the case of homoskedastic errors. This paper, and Ozabaci et al. (2014), impose additivity on $h(X)$ in an effort to realize these efficiency gains while Geng et al. (2016) do not. Secondly, as pointed out in Linton and Neilsen (1995), and Buja et al. (1989), additivity allows one to avoid the "curse of dimensionality" while sacrificing little in terms of model flexibility. This "curse" refers to the case where the sum of univariate nonparametric estimators converges at a rate equal to the slowest univariate estimator, while the rate of convergence of a multivariate estimator is inversely proportional to the dimension of its argument. In the sense of avoiding the curse of dimensionality one can draw distinctions between this paper, Geng et al. (2016), and Ozabaci et al. (2014). Given that this paper and Geng et al. (2016) both develop estimators for $[\beta_0 \;\; \beta_1']'$ separate from the estimation $h(X)$, the potential rate of convergence of $h(X)$ is irrelevant where it is of primary interest to Ozabaci et al. (2014). However imposing additivity on $m(X)$ is of consequence to all three models as each relies on the estimation of the residual vector $\hat{V} = X - \hat{m}(W)$ in the estimation of the primary equation. This paper and Ozabaci et al. (2014), avoid the "curse" by specifying $m(X)$ to be additive while Geng et al. (2016) do not. The importance of these restrictions manifest themselves in Theorem 3 where I show that the asymptotic covariance matrix of the estimator presented in this paper is unambiguously smaller than that of Geng et al. (2016). In fact my estimator is Oracle efficient in the sense that the covariance matrix derived in this paper is equal to the covariance matrix of the semiparametrically efficient estimator of Manzan and Zerom (2005), indicating that asymptotically there is no penalty to estimating $\hat{V}$ as a preliminary step in the estimation of $[\beta_0 \;\; \beta_1']'$. I refer to this estimator as Oracle efficient in a sense conceptually similar to, inter alia, Yu et al. (2008), and Horowitz and Mammen (2004). These papers describe the estimation of a collection of functions to be Oracle efficient if the asymptotic covariance matrix of an estimator for each component is the same as if all other components in the model are known functions. This paper details the estimation of a collection of functions, that I call Oracle efficient since the asymptotic covariance of my estimator for β_1 is the same as if the other components to be estimated $m(W)$ are known functions.

In terms of estimation, both this paper and Geng et al. (2016) are based on the identification and estimation procedure given in Manzan and Zerom (2005) requiring at least two steps for estimation. In the first step estimation of $m(X)$, Geng et al. (2016) use a kernel based Nadaraya Watson estimator, where I use an additive B spline series estimator. In subsequent steps both Geng et al. (2016) and this paper use kernel based estimators. In this way Geng et al. (2016) is a fully kernel based estimator where the estimator in this paper is a mixture of series and kernel methods. Lastly due to the fully nonparametric additive form of their model, Ozabaci et al. (2014) use two stages of series estimation. The results of the Monte Carlo simulation presented in section 4 show that the estimator in this paper outperforms the estimator of Geng et al. (2016), in all cases. In addition the Monte Carlo results show that even in finite samples that properties of the estimator in this paper are very similar to those of the "Oracle" estimator of Manzan and Zerom (2005) suggesting that even in finite samples there is very little penalty to estimating $m(W)$.

Beyond the papers mentioned above there are two other alternative estimators accounting for endogeneity in a semiparametric setting; Ai and Chen (2003), and Otsu (2011). What distinguishes Ai and Chen (2003), and Otsu (2011) from Geng et al. (2016), Ozabaci et al. (2014), and this paper are the moment conditions upon which they are based. The sieve minimum distance estimator of Ai and Chen (2003) and the empirical likelihood estimator of Otsu (2011) both make the assumption that $E(\varepsilon|W) = 0$ in contrast to $E(\varepsilon|W, X) = E(\varepsilon|V)$ of equation (3). As discussed in Newey et al. (1999) neither set of moment conditions is stronger than the other since (3) does not imply $E(\varepsilon|W) = 0$ without additional assumptions. These estimation procedures are shown to be semiparametrically efficient but are numerical procedures with no closed form solution and are extremely difficult to apply empirically where the estimator developed in this paper is straightforward to apply.

The remainder of this paper is organized into 4 Sections; Section 2 details the relevant moment conditions, identification, and estimation of my model. Section 3 gives all necessary assumptions, and summarizes the asymptotic results for each step of the estimation procedure. Section 4 presents a Monte Carlo study of this estimation procedure that demonstrates. Lastly section 5 is the appendix where proofs of all Theorems and supporting Lemmas are given.

2 Moment Conditions, Identification, and Estimation

2.1 Moment Conditions

$E(\varepsilon|X) \neq 0$ implies that ε is not orthogonal to the space of square integrable functions of X. The approach taken to deal with this lack of orthogonality is a variation on the control function approach of Newey et al. (1999). In particular, defining $u = \varepsilon - f(V)$, I assume,

$$E[\varepsilon|Z,X,V] = E[\varepsilon|V] = E[f(V)+u|V] = f(V) + E[u|V] = f(V), \qquad \text{and} \qquad E[u|W,X,V] = 0. \tag{4}$$

Furthermore, I assume,

$$h(X) = \sum_{d=1}^{D} h_d(X_d), \qquad f(V) = \sum_{d=1}^{D} f_d(V_d), \quad \text{and} \quad m_d(W) = \sum_{a=1}^{q} m_{da}(W_a). \tag{5}$$

Consequently,

$$Y = \beta_0 + Z'\beta_1 + \sum_{d=1}^{D} h_d(X_d) + \sum_{d=1}^{D} f_d(V_d) + u, \tag{6}$$

$$X_d = \sum_{a=1}^{q} m_{da}(W_a) + V_d, \tag{7}$$

$$E(V_d|W) = 0, \quad E(\varepsilon|W,X,V) = 0. \tag{8}$$

The goal of this paper is to identify and estimate the parameters $[\beta_0\ \beta_1']' \in \mathbb{R}^{p+1}$ in such a way that neither procedure requires any restriction on the functions $h(X)$ and $f(V)$ beyond the standard identification condition that $E(h_d(X_d)) = 0$ and $E(f_d(V_d)) = 0$. The identification detailed in the following section is a variation of the identification procedure developed in Manzan and Zerom (2005).

2.2 Identification

Let $p(\cdot)$ be the marginal/joint densities (provided they exist) of its random variable/vector argument and define,

$$g(X,V) = \prod_{j=1}^{D} p(X_j) \prod_{k=1}^{D} p(V_k), \qquad g(X_{-d},V) = \prod_{j\neq d}^{D} p(X_j) \prod_{k=1}^{D} p(V_k), \qquad g(X,V_{-d}) = \prod_{j=1}^{D} p(X_j) \prod_{k\neq d}^{D} p(V_k).$$

Furthermore as in Kim et al. (1999) and Geng et al. (2016) define an "instrument function" $\phi \equiv \phi(X,V) \equiv g(X,V)/p(X,V)$ and related functions, $\theta_1^d \equiv \theta_1^d(X,V) \equiv g(X_{-d},V)/p(X,V)$ and $\theta_2^d \equiv$

$\theta_2^d(X,V) \equiv g(X,V_{-d})/p(X,V)$. Now, let A be any subvector of $[\ Y\ \ Z'\ ]'$, let $\mu_A = E[\phi A]$, and define

$$H_1^d(A) = E[\phi A|X_d], \qquad H_2^d(A) = E[\phi A|V_d], \tag{9}$$

$$H(A) = \sum_{d=1}^{D}\Big[H_1^d(A) + H_2^d(A)\Big], \qquad H^*(A) = H(A) - (2D-1)\mu_A. \tag{10}$$

Furthermore, for all $c \in \{1,2,\ldots,p\}$ define the following differences,

$$\zeta_c \equiv Z_c - H^*(Z_c), \qquad \zeta \equiv Z - H^*(Z), \tag{11}$$

$$\rho_c \equiv Z_c - E[Z_c|X,V], \qquad \rho \equiv Z - E[Z|X,V], \tag{12}$$

$$\eta_{1c}^d \equiv E[Z_c|X,V] - H_1^d(Z_c), \qquad \eta_{2c}^d \equiv E[Z_c|X,V] - H_2^d(Z_c), \tag{13}$$

$$\eta \equiv E[Z|X,V] - H^*(Z). \tag{14}$$

The following lemma gives conditions for the identification of $[\beta_0\ \ \beta_1']' \in \mathbb{R}^{p+1}$.

Lemma 1. *Parameters β_0 and β_1 are identified if either of the following sets of condition are satisfied.*

i) $E(h_d(X_d)) = 0$, $E(f_d(V_d)) = 0$ *for all* $d \in \{1,2,\ldots,D\}$ *and either*

- *a)* $E(\rho\rho') > 0$, *or*
- *b)* $E(\eta\eta') > 0$.

ii) $E(h_d(X_d)) = 0$, $E(f_d(V_d)) = 0$ *for all* $d \in \{1,2,\ldots,D\}$ *and,* $E(\phi\zeta\zeta') > 0$

In addition to the conditions under which β_0 and β_1 are identified, I now state moment conditions implied by identification, which are critical to the proof of $\sqrt{n}$ convergence of the estimators for $[\beta_0, \beta_1']'$.

$$E[\phi\zeta_c|X_d] = 0, \qquad E[\phi\zeta_c|V_d] = 0, \tag{15}$$

$$E[\phi(Y - H^*(Y))|X_d] = 0, \qquad E[\phi(Y - H^*(Y))|V_d] = 0. \tag{16}$$

2.3 Estimation

Motivated by the moment and identification conditions given in the previous sub-section, and the assumption that $\{Y_i, X_i, W_i\}_{i=1}^n$ is an i.i.d sequence (see Assumption A3), I now describe the estimation procedure.

Step One: Obtain a B-Spline series estimate of $m(W_i)$ with dth element

$$\hat{m}_d(W_i) = \mathbf{B}_n(W_i)'(\mathbf{B}_n'\mathbf{B}_n)^{-1}\mathbf{B}_n'\mathbf{X}_{dn}. \tag{17}$$

where, $\mathbf{B}_n(W_i) = \left[\mathbf{B}_n^{l_n}(W_{1i}) \;\; \mathbf{B}_n^{l_n}(W_{2i}) \;\; \cdots \;\; \mathbf{B}_n^{l_n}(W_{qi})\right]'$ is a $(q(l_n + 2k) \times 1)$ vector of B-spline basis functions, $\mathbf{B}_n = [\mathbf{B}_n(W_1) \;\; \mathbf{B}_n(W_2) \;\; \cdots \;\; \mathbf{B}_n(W_n)]'$, and $\mathbf{X}_{dn} = [X_{d1} \;\; X_{d2} \;\; \cdots \;\; X_{dn}]'$. These estimates will then be used to obtain residuals,

$$\hat{V}_{di} = X_{di} - \hat{m}_d(W_i), \qquad \text{and define} \qquad \hat{\mathbf{V}}_{dn} = \left[\; \hat{V}_{d1} \;\; \hat{V}_{d2} \;\; \cdots \;\; \hat{V}_{dn} \;\right]'. \tag{18}$$

Step Two: Obtain Rosenblatt kernel density estimates of $p(X_i, V_i), p(X_{di})$, and $p(V_{di})$ using observed values X_i and estimated values $\hat{V}_i$. Let $\mathbf{1}_D$ be a $(D \times 1)$ vector of ones, and define $H = diag([h_1 \cdot \mathbf{1}_D', h_2 \cdot \mathbf{1}_D]')$,

$$\hat{p}(X_{di}) = \frac{1}{nh_0} \sum_{l=1}^{n} K_0\left(h_0^{-1}[X_{dl} - X_{di}]\right), \qquad \hat{p}(\hat{V}_{di}) = \frac{1}{nh_0} \sum_{l=1}^{n} K_0\left(h_0^{-1}[\hat{V}_{dl} - \hat{V}_{di}]\right), \tag{19}$$

$$\hat{p}(X_i, \hat{V}_i) = \frac{1}{nh_1^D h_2^D} \sum_{l=1}^{n} K_3\big(H^{-1}[(X_l', \hat{V}_l')' - (X_i', \hat{V}_i')']\big), \tag{20}$$

where $K_0(\cdot) : \mathbb{R} \to \mathbb{R}$ is a univariate Kernel with bandwidth h_0. $K_3(\cdot) : \mathbb{R}^{2D} \to \mathbb{R}$ is a multivariate kernel with associated bandwidths h_1 corresponding to the X arguments and h_2 corresponding to the $\hat{V}$ arguments. Next define,

$$\hat{g}(X_i, \hat{V}_i) = \prod_{d=1}^{D} \hat{p}(X_{di})\hat{p}(\hat{V}_{di}), \quad \hat{g}(X_{-di}, \hat{V}_i) = \prod_{j \neq d}^{D} \hat{p}(X_{ji}) \prod_{k=1}^{D} \hat{p}(\hat{V}_{ki}), \quad \hat{g}(X_i, \hat{V}_{-di}) = \prod_{j=1}^{D} \hat{p}(X_j) \prod_{k \neq d}^{D} \hat{p}(\hat{V}_k),$$

with which I obtain $\hat{\phi}_i \equiv \hat{g}(X_i, \hat{V}_i)/\hat{p}(X_i, \hat{V}_i)$, $\hat{\theta}_{1i}^d \equiv \hat{g}(X_{-di}, \hat{V}_i)/\hat{p}(X_i, \hat{V}_i)$, and $\hat{\theta}_{2i}^d \equiv \hat{g}(X_i, \hat{V}_{-di})/\hat{p}(X_i, \hat{V}_i)$.

Step Three : Obtain Nadaraya Watson estimates of conditional expectation's $H_1^d(Z_{ci})$, $H_2^d(Z_{ci})$, $H_1^d(Y_i)$, and $H_2^d(Y_i)$, i.e.

$$\hat{H}_1^d(Z_{ci}) = [(n-1)b_1]^{-1} \sum_{l \neq i}^{n} K_1[b_1^{-1}(X_{dl} - X_{di})]\hat{\theta}_{1l}^d Z_{cl}, \tag{21}$$

$$\hat{H}_2^d(Z_{ci}) = [(n-1)b_2]^{-1} \sum_{l \neq i}^{n} K_2[b_2^{-1}(\hat{V}_{dl} - \hat{V}_{di})]\hat{\theta}_{2l}^d Z_{cl}, \tag{22}$$

$$\hat{H}_1^d(Y_i) = [(n-1)b_1]^{-1} \sum_{l \neq i}^{n} K_1[b_1^{-1}(X_{dl} - X_{di})]\hat{\theta}_{1l}^d Y_l, \tag{23}$$

$$\hat{H}_2^d(Y_i) = [(n-1)b_2]^{-1} \sum_{l \neq i}^{n} K_2[b_2^{-1}(\hat{V}_{dl} - \hat{V}_{di})]\hat{\theta}_{2l}^d Y_l, \tag{24}$$

where $K_1(\cdot), K_2(\cdot) : \mathbb{R} \to \mathbb{R}$ are univariate Kernels with bandwidths b_1 and b_2 respectively. Additionally,

$\mu(Z_c)$, and μ_Y are estimated with $\hat{\mu}_{Z_c} = n^{-1}\sum_{i=1}^{n}\hat{\phi}_i Z_{ci}$ and $\hat{\mu}_Y = n^{-1}\sum_{i=1}^{n}\hat{\phi}_i Y_i$ so that,

$$\hat{H}(Z_{ci}) = \sum_{d=1}^{D}[\hat{H}_1^d(Z_{ci}) + \hat{H}_2^d(Z_{ci})], \qquad \hat{H}(Y_i) = \sum_{d=1}^{D}[\hat{H}_1^d(Y_i) + \hat{H}_2^d(Y_i)], \tag{25}$$

$$\hat{H}^*(Z_{ci}) = \hat{H}(Z_{ci}) - (2D-1)\hat{\mu}_{Z_c}, \qquad \hat{H}^*(Y_i) = \hat{H}(Y_i) - (2D-1)\hat{\mu}_Y. \tag{26}$$

Let $\hat{\mathbf{H}}_n^*(Y) = [\ \hat{H}^*(Y_1) \quad \hat{H}^*(Y_2) \quad \cdots \quad \hat{H}^*(Y_n)\ ]'$, $\hat{\mathbf{H}}_n^*(Z_c) = [\ \hat{H}^*(Z_{c1}) \quad \hat{H}^*(Z_{c2}) \quad \cdots \quad \hat{H}^*(Z_{cn})\ ]'$ and $\hat{\mathbf{H}}_n^*(Z) = [\ \hat{\mathbf{H}}_n^*(Z_1) \quad \hat{\mathbf{H}}_n^*(Z_2) \quad \cdots \quad \hat{\mathbf{H}}_n^*(Z_p)\ ]$ where $\mathbf{Y}_n = [\ Y_1 \quad Y_2 \quad \cdots \quad Y_n\ ]'$, and $\mathbf{Z}_n = [Z_1 \quad Z_2 \quad \cdots \quad Z_n\ ]'$ and let,

$$\hat{\zeta}_{ci} = Z_{ci} - \hat{H}^*(Z_{ci}), \qquad \hat{\zeta}_i = Z_i - \hat{H}^*(Z_i), \qquad \hat{\boldsymbol{\zeta}}_n = \mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z). \tag{27}$$

Step Four: Estimation of $(\ \beta_0 \quad \beta_1'\ )' \in \mathbb{R}^{p+1}$.
I regress $\hat{\boldsymbol{\phi}}_n^{1/2}(\mathbf{Y}_n - \hat{\mathbf{H}}_n^*(Y))$ on $\hat{\boldsymbol{\phi}}_n^{1/2}\hat{\boldsymbol{\zeta}}_n = \hat{\boldsymbol{\phi}}_n^{1/2}(\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z))$ and estimate β_1 as,

$$\hat{\beta}_1 = [\hat{\boldsymbol{\zeta}}_n'\hat{\boldsymbol{\phi}}_n\hat{\boldsymbol{\zeta}}_n]^{-1}\hat{\boldsymbol{\zeta}}_n'\hat{\boldsymbol{\phi}}_n(\mathbf{Y}_n - \hat{\mathbf{H}}_n^*(Y)). \tag{28}$$

where $\hat{\boldsymbol{\phi}}_n \equiv \hat{\boldsymbol{\phi}}_n(X, \hat{V}) = diag(\{\hat{\phi}(X_i, \hat{V}_i)\}_{i=1}^n)$, Lastly β_0 is estimated by,

$$\hat{\beta}_0 = \hat{\mu}_Y - \hat{\mu}_Z'\hat{\beta}_1. \tag{29}$$

3 Asymptotic Results

In this section, I study the asymptotic properties of $[\hat{\beta}_0\ \hat{\beta}_1']'$. To do so, I first define vector and matrix norms that I will use throughout this paper. For $m, n \in \mathbb{N}$, let A be a matrix of order $(m \times n)$, r be a random vector of order $(m \times 1)$ whose elements have finite second moments, and c be a non stochastic vector of order $(m \times 1)$. Define $||A|| = \sqrt{trace(AA')}$, the Frobenius matrix norm, define the matrix spectral norm $||A||_{sp} = \max_{1 \le i \le m} \lambda_i(AA')^{1/2}$ where $\{\lambda_i(AA')\}_{i=1}^m$ the collection of all eigenvalues of AA'. Define the inner product $||r||_2^2 = E(r'r)$, and lastly define $||c||_E = (c'c)^{1/2}$ the typical Euclidean norm in finite dimensional space $\mathbb{R}^m$.

3.1 Assumptions

Assumption A1: The B-spline basis functions used in step one satisfy the following: (i) Let $\{l_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of natural numbers such that $l_n^2/n = o(1)$. Let $G_{W_a} \subset \mathbb{R}$ be the compact support of W_a, and without loss of generality. let $G_{W_a} = [0, 1]$. For some $k \ge 3$ let $\{t_j\}_{j=1}^{l_n+2k}$ be a knot set [2] for G_{W_a}. Now define the following functions

$$b_{j,1}(W_a) = \begin{cases} 1 & \text{if } W_a \in [t_j, t_{j+1}) \\ 0 & \text{if } W_a \notin [t_j, t_{j+1}), \end{cases} \qquad \text{and} \qquad \omega_{j,k}(W_a) = \frac{W_a - t_j}{t_{j+k-1} - t_j}.$$

Now, define the normalized B-spline basis of order $k \in \mathbb{N}$ functions recursively,

$$b_{j,k}(W_a) = \omega_{j,k}(W_a) b_{j,k-1}(W_a) + (1 - \omega_{j+1,k}(W_a)) b_{j+1,k-1}(W_a) \qquad \text{and} \qquad B_{j,k}(W_a) = \frac{b_{j,k}(W_a)}{||b_{j,k}(W_a)||_2}$$

where $\{B_{j,k}(W_a)\}_{j=1}^{l_n+2k}$ is a sequence of uniformly bounded, positive, twice integrable, compactly supported B-spline basis functions with which I construct,

$$\mathbf{B}_n(W_{ai})' = \begin{bmatrix} B_{1,k}(W_{ai}) & \cdots & B_{l_n+2k,k}(W_{ai}) \end{bmatrix}$$

(ii) Define $Q_{BB} = E\left[\mathbf{B}_n(W_i)\mathbf{B}_n(W_i)'\right]$, $Q_{nBB} = n^{-1}\sum_{i=1}^{n} \mathbf{B}_n(W_i)\mathbf{B}_n(W_i)'$, $Q^d_{BBV} = E\left[\mathbf{B}_n(W_i)\mathbf{B}_n(W_i)'V_{di}^2\right]$, and $Q^d_{nBBV} = n^{-1}\sum_{i=1}^{n} \mathbf{B}_n(W_i)\mathbf{B}_n(W_i)'V_{di}^2$. Assume that for n sufficiently large, there exist constants $c_{Bl}, c_{Bh}, c_{BV} \in \mathbb{R}^+$ such that

$$0 < c_{Bl} \le \lambda_{min}(Q_{BB}) \le \lambda_{max}(Q_{BB}) \le c_{Bh} < \infty \qquad \text{and} \qquad 0 < \lambda_{max}(Q^d_{BBV}) \le c_{BV} < \infty$$

Remarks : Under these assumptions, by Theorem (6) de Boor (2001), for every n, there exists a parameter vector $\alpha^d_{l_n} \in \mathbb{R}^{l_n+2k}$ and a function $m_d^{l_n}(W_{ai}) = \mathbf{B}^{l_n}(W_{ai})'\alpha_d^{l_n}$ such that,

$$\sup_{W_a \in G_{W_a}} |m_d(W_a) - m_d^{l_n}(W_a)| = O\left(l_n^{-k}\right)$$

Additionally, it should be clear that for any finite k, $B_{jk}(W_a)$ has shrinking compact support [3] as $l_n \to \infty$. In particular,

$$B_{j,k}(W_a) \begin{cases} > 0 & \text{if } W_a \in [t_j, t_{j+k}) \\ = 0 & \text{if } W_a \notin [t_j, t_{j+k}). \end{cases}$$

Assumption A2: For all, $j \in 0,1,2,3$ (i) $K_j(\gamma) : \mathbb{R} \to \mathbb{R}$ is 4 times continuously differentiable, (ii) $K_j(\gamma)$ is symmetric about zero (iii) $K_j(\gamma)$ is a kernel of order ν_j where $\int K_j(\gamma)\gamma^\alpha d\gamma = 0$ for all $\alpha = 1, 2, \ldots, \nu_j - 1$ and $\int |K_j(\gamma)||\gamma|^{\nu_j} d\gamma < \infty$ (iv) $\int K_j(\gamma) d\gamma = 1$, (v) $|K_j^{(\alpha)}(\gamma)||\gamma|^{5+a} \to 0$ as $|\gamma| \to \infty$ for

[2] See de Boor (2001) for details of knot set construction
[3] See de Boor (2001)

some $a > 0$, where $K_j^{(\alpha)}(\gamma)$ is the αth derivative of K_j. (vi) $\int |K_j^{(1)}(\gamma)||\gamma|^{\nu_j} d\gamma$ (vii) for $2 \leq m \leq \nu_j - 1$, $\int K_j^{(1)}(\gamma)^2 \gamma^{2m} d\gamma$. $(viii)$ Let $\gamma_1 = [\gamma_{11}\ \gamma_{12}\ \cdots\ \gamma_{1D}]' \in \mathbf{R}^D$, $\gamma_2 = [\gamma_{21}\ \gamma_{22}\ \cdots\ \gamma_{2D}]' \in \mathbf{R}^D$ and define,

$$K_3([\gamma_1', \gamma_2']') \equiv \prod_{d=1}^{D} K_3(\gamma_{1d}) \prod_{d=1}^{D} K_3(\gamma_{2d}).$$

Assumption A3: $(i.)$ $\{Y_i, X_i, W_i\}_{i=1}^n$ is an i.i.d sequence of random vectors distributed as (Y, X, W) with finite first and second moments. $(ii.)$ Densities; $p(W)$, $p(X_d)$, $p(V_d)$, $p(X, V)$ and $p(X, V, W)$ exist and are bounded away from zero and infinity on their convex and compact supports. (iii) The parameter vector $[\ \beta_0\ \ \beta_1'\ ]'$ is an element of a bounded (w.r.t. the Euclidean norm) subset of $\mathbb{R}^{p+1}$.
Remarks: (ii) implies that ϕ_i, θ_{2i}^d, and θ_{2i}^d are uniformly bounded away from zero and infinity.

Assumption A4: Let $\alpha, b \in \mathbb{N}$ and define $\mathcal{F}_b$ as a class of real valued $f : \mathbb{R}^\alpha \to \mathbb{R}$ functions s.t. if $f \in \mathcal{F}_b$ then: f is everywhere $b \in \mathbb{N}$ times continuously partially (provided $\alpha > 1$) differentiable. f and all of its derivatives are uniformly bounded. $(i.)$ Let, $p(W), p(X_d), p(V_d) \in \mathcal{F}_{\nu_0}$, and $p(X, V), p(X, V, W) \in \mathcal{F}_{\nu_3}$. $(ii.)$ $H_1^d(Z_c)$, $H_2^d(Z_c)$, $H_1^d(Y)$, $H_2^d(Y)$, $E[Z_c|X, V]$, $E[Y|X, V] \in \mathcal{F}_{\nu_{12}}$, where $\nu_{12} = \max(\nu_1, \nu_2)$.
Remarks: (i) is necessary for a sufficiently fast uniform rate of convergence of the Rosenblatt kernel density estimators generated in step 2. (ii) is necessary for various Taylor expansions of those functions in the proofs of the Theorems and Lemmas.

Assumption A5: (i) $E(h_d(X_d)) = 0$, $E(f_d(V_d)) = 0$, and $\Sigma_1 \equiv E(\phi\zeta\zeta')$ is a positive definite matrix. (ii) $E(\zeta_c^2) < \infty$, $E(u^2|Z, X, V) = \sigma_u^2 < \infty$. (iii) Functions; $E(\rho_c^2|X_d)$, $E(\rho_c^2|V_d)$, $E([\eta_{1c}^d]^2|V_d)$, $E([\eta_{2c}^d]^2|X_d)$, $E(\phi^2\zeta_c^2|X_d)$, $E(\phi^2\zeta_c^2|V_d)$, and $E(\phi^2\zeta_c^2|W)$ are uniformly bounded on their compact supports.
Remarks: (i) Ensures identification of $[\beta_o\ \ \beta_1']'$. (ii) Provides upper bounds for conditional moment used throughout the proofs of Theorems 1-3 , along with providing for the uniform convergence of several terms in accordance with the results of Lemma 6.

Assumption A6: Let $a, b, c, e, f, C \in \mathbb{R}^+$, $\nu_0, \nu_1, \nu_2, \nu_3 \in \mathbb{N}$ and $k \geq 3$ as defined in assumption 1. If,

$$l_n = Cn^a, \qquad b_1 = Cn^{-e}, \qquad b_2 = Cn^{-b}, \qquad h_0 = Cn^{-f}, \qquad h_1 = Cn^{-c}, \qquad h_2 = Cn^{-c}.$$

Then constants $\{a, b, c, e, f, \nu_0, \nu_1, \nu_2\}$ satisfy the following (i) $1/4 > a > 1/2k$, (ii) $b < 1/4 - a$, (iii) $c < \min\Big(1/4(D+1), (2k-1)/(4k(D+1)), (3k-3)/16k\Big)$ (iv) $e < 1/2$, (v) $f < 1/2$, (vii) For some arbitrarily small $\varepsilon > 0$

$$\nu_0 = \inf\{x \in \mathbb{N} : x > 1/4f\},$$
$$\nu_1 = \inf\{x \in \mathbb{N} : x > 1/4e\},$$
$$\nu_2 = \inf\{x \in \mathbb{N} : x > 1/4b\},$$
$$\nu_3 = \inf\left\{x \in \mathbb{N} : x > \left(4\left[\min\left(\frac{1}{4(D+1)}, \frac{2k-1}{4k(D+1)}, \frac{3k-3}{16k}\right) - \varepsilon\right]\right)^{-1}\right\}.$$

Remarks : Assumptions (i) - (vi), are sufficient conditions for the rates of convergence derived in Lemma 3. Note that the conditions on h_0, b_1, ν_0, and ν_1 are set to ensure that uniform rates of convergence of kernel estimators used in steps two through four are at least $o_p(n^{-1/4})$. These constitute minimum requirements thus, if possible, some constants may be changed to improve a upon a minimum rate of $o_p(n^{-1/4})$.

3.2 Theorems

For any collection of random variables $\{M_j\}_{j=1}^J$ let G_M be the cartesian product of the support of each real random variable. By Theorem 1 in Newey (1997), under the Assumption A1 of this paper,

$$\sup_{W_a \in G_{W_a}} |\hat{m}_{da}(W_a) - m_{da}(W_a)| = O_p\left(\frac{l_n}{\sqrt{n}} + l_n^{1/2-k}\right) \equiv O_p(L_n).$$

Consequently, since $\hat{V}_{di} - V_{di} = X_{di} - \hat{m}_d(W_i) - X_{di} + m_d(W_i) = \sum_{a=1}^q \left(\hat{m}_d(W_{ai}) - m_d(W_{ai})\right)$ I have,

$$\sup_{W \in G_W} |\hat{V}_d - V_d| \leq \sum_{a=1}^{q} \sup_{W_a \in G_{W_a}} |\hat{m}_{da}(W_a) - m_{da}(W_a)| = O_p(L_n). \tag{30}$$

For notational simplicity set,

$$M_{1n} \equiv \left[\frac{\log(n)}{nh_0}\right]^{1/2} + h_0^{\nu_0}, \qquad M_{2n} \equiv \left[\frac{\log(n)}{nh_1^D h_2^D}\right]^{1/2} + h_1^{\nu_3} + h_2^{\nu_3}, \qquad M_n \equiv M_{1n} + M_{2n}, \tag{31}$$

$$N_{1n} \equiv \left[\frac{\log(n)}{nb_1}\right]^{1/2} + b_1^{\nu_1}, \qquad N_{2n} \equiv \left[\frac{\log(n)}{nb_2}\right]^{1/2} + b_2^{\nu_2}, \qquad \mathcal{L}_{0n} \equiv L_n + M_n, \tag{32}$$

$$\mathcal{L}_{1n} \equiv L_n + M_n + N_{1n}, \qquad \mathcal{L}_{2n} \equiv L_n + M_n + N_{2n} + b_2^{-1}\left[\sqrt{\frac{l_n}{n}} + l_n^{-k}\right], \qquad \mathcal{L}_n \equiv \mathcal{L}_{1n} + \mathcal{L}_{2n}. \tag{33}$$

The following Theorem shows that the uniform rate of convergence of $\hat{p}(\hat{V}_{di})$ to $p(V_{di})$, is equal to minimum of uniform rates of convergence of either $\hat{m}_d(W_{ai})$ to $m_d(W_{ai})$, or $\hat{p}(V_{di})$ to $p(V_{di})$. Likewise the uniform rate of convergence of $\hat{p}(X_i, \hat{V}_i)$ to $p(X_i, V_i)$ is equal to minimum of uniform rates of convergence of either $\hat{m}_d(W_{ai})$ to $m_d(W_{ai})$, $O_p(L_n)$ or $\hat{p}(X_i, V_i)$ to $p(X_i, V_i)$, $O_p(M_{2n})$. Thus clearly if both M_{1n}

and M_{2n} dominate L_n the rate of convergence derived in Theorem 1 is the same as if $m(W)$ is a known function. Furthermore the same holds true for the uniform rate of convergence of density ratios $\hat{\phi}_i$, $\hat{\theta}^d_{1i}$, and $\hat{\theta}^d_{2i}$.

Theorem 1. *Under Assumptions A1 - A5,*

$$\sup_{X_{di}\in G_{X_d}} |\hat{p}(X_{di}) - p(X_{di})| = O_p(M_{1n}), \qquad \sup_{V_{di}\in G_{V_d}} |\hat{p}(\hat{V}_{di}) - p(V_{di})| = O_p(L_n + M_{1n}),$$

$$\sup_{X_i,V_i\in G_{X,V}} |\hat{p}(X_i,\hat{V}_i) - p(X_i,V_i)| = O_p(L_n + M_{2n}), \qquad \sup_{X_i,V_i\in G_{X,V}} |\hat{\phi}(X_i,\hat{V}_i) - \phi(X_i,V_i)| = O_p(\mathcal{L}_{0n}),$$

$$\sup_{X_i,V_i\in G_{X,V}} |\hat{\theta}(X_{-di},\hat{V}_i) - \theta(X_{-di},V_i)| = O_p(\mathcal{L}_{0n}), \qquad \sup_{X_i,V_i\in G_{X,V}} |\hat{\theta}(X_i,\hat{V}_{-di}) - \theta(X_i,V_{-di})| = O_p(\mathcal{L}_{0n}).$$

Theorem 2. *Under assumptions A1 - A6, $\hat{\mu}_Y - \mu_Y = O_p(\mathcal{L}_{0n})$, $\hat{\mu}(Z_c) - \mu(Z_c) = O_p(\mathcal{L}_{0n})$ and*

$$\sup_{X_{di}\in G_{X_d}} |\hat{H}^d_1(Y_i) - H^d_1(Y_i)| = O_p(\mathcal{L}_{1n}), \qquad \sup_{V_{di}\in G_{V_d}} |\hat{H}^d_2(Y_i) - H^d_2(Y_i)| = O_p(\mathcal{L}_{2n}),$$

$$\sup_{X_{di}\in G_{X_d}} |\hat{H}^d_1(Z_{ci}) - H^d_1(Z_{ci})| = O_p(\mathcal{L}_{1n}), \qquad \sup_{V_{di}\in G_{V_d}} |\hat{H}^d_2(Z_{ci}) - H^d_2(Z_{ci})| = O_p(\mathcal{L}_{2n}),$$

$$\sup_{X_i,V_i\in G_{X,V}} |\hat{H}^*(Y_i) - H^*(Y_i)| = O_p(\mathcal{L}_n), \qquad \sup_{X_i,V_i\in G_{X,V}} |\hat{H}^*(Z_{ci}) - H^*(Z_{ci})| = O_p(\mathcal{L}_n).$$

Remarks : The uniform rate of convergence of these estimators are the sum of the uniform rates of convergence of $\hat{m}_d(W_{ai})$, $\hat{\theta}(X_{-di},\hat{V}_i)$ or $\hat{\theta}(X_i,\hat{V}_{-di})$, and a typical NW estimator, in each case except for those terms where the conditional expectation is taken with respect to (w.r.t) V. In this case there is an additional term $b_2^{-1}\big([l_n/n]^{1/2} + l_n^{-k}\big)$ which results from having to take a Taylor expansion of the kernel $K_2(\cdot)$ evaluated at $V_{dl} - V_{di}$. This additional term does slow down the rate of convergence. However, all that is required is $\mathcal{L}_{0n} = o(n^{-1/4})$ which, as shown in Lemma 3, is accomplished under Assumption 6 . In the following I present the main theorem of this paper, $\sqrt{n}$ asymptotic normality of $\hat{\beta}_1 - \beta_1$,

Theorem 3. *Under assumptions A1 - A6,*

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} N(0, \Sigma_0^{-1}\Sigma_1\Sigma_0^{-1})$$

where matrices Σ_0, Σ_1 have typical elements $\Sigma_0(a,c) = E\left[(Z_{ai} - H^*(Z_{ai})\phi(X_i,V_i)(Z_{ci} - H^*(Z_{ci})\right]$, and $\Sigma_1(a,c) = E\left[(Z_{ai} - H^*(Z_{ai})\phi(X_i,V_i)^2(Z_{ci} - H^*(Z_{ci})\right]\sigma_u^2$ where $a,c \in \{1,2,\dots,p\}$

Remarks: Note that $\hat{\beta}_1$ is $\sqrt{n}$ asymptotically normal with a covariance matrix equal to an identical estimation procedure where $m(W)$ is a known function. This clearly indicates that asymptotically there is no penalty, in terms of variance, to estimating $m(W)$ rather than observing it. Furthermore the asymptotic variance of $\hat{\beta}_1$ is the same as the asymptotic variance derived by Manzan and Zerom (2005) which, as they point out in the case of homoskedastic errors, meets the efficiency bound derived by Chamberlain (1992) for their model, where $m(X)$ is observed. However I can make no claim to efficiency for my model since such an efficiency bound does not currently exist in the statistical literature.

4 Monte Carlo Study

In this section I investigate the finite sample performance of $[\hat{\beta}_1, \hat{m}(W)]$ and compare it to the finite sample performance of both $[\hat{\beta}_{1,G}, \hat{m}_G(W)]$, and $[\hat{\beta}_{1,M}, m(W)]$ the estimator presented in Geng et al. (2016), and the estimator of Manzan and Zerom (2005) where $m(W)$ is taken to be a known vector of functions, respectively. I illustrate the differences between these estimators in terms of two data generating processes DGP $\in \{1, 2\}$, three values for a parameter $\theta_e \in \{0.3, 0.6, 0.9\}$ indicating an increasing strength of the endogenous relationship between X and ε, and two sample sizes $n \in \{200, 400\}$. Consider the following DGPs whose leading non nonlinear terms are the same, at least in part, as the functions used in Geng et al. (2016), Su and Ullah (2008), Martins-Filho and Yao (2012), and Ai and Chen (2003),

$$\text{DGP} = 1: \quad Y_i = \beta_0 + Z_{1i}\beta_1 + \log(|X_{1i} - 1| + 1)\text{sgn}(X_{1i} - 1) - X_{2i} + \left[\frac{X_{2i}}{3}\right]^2 + \varepsilon_i,$$
$$\text{DGP} = 2: \quad Y_i = \beta_0 + Z_{1i}\beta_1 + \frac{\exp(X_{1i})}{1 + 3\exp(X_{1i})} - \left[\frac{X_{2i}}{3}\right]^2 + \varepsilon_i,$$

where I set $\beta_0 = 1$ and $\beta_1 = 10$. Both DGPs have identical secondary equations,

$$X_{1i} = 1/2Z_{1i}^2 - 1/2Z_{2i}^2 - 2Z_{3i} + V_{1i}, \quad \text{and} \quad X_{2i} = 3/2Z_{1i} + 1/2Z_{2i}^2 + 1/3Z_{3i}^3 + V_{2i},$$

where $\{W_i\}_{i=1}^n = \{Z_{1i}, Z_{2i}, Z_{3i}\}_{i=1}^n$ is an i.i.d sequence with distribution $W \sim N(0, I_3)$ and $\{\varepsilon_i, V_{1i}, V_{2i}\}_{i=1}^n$ is an i.i.d sequence with distribution,

$$\begin{pmatrix} \varepsilon_i \\ V_{1i} \\ V_{2i} \end{pmatrix} \sim N\left(0, \begin{pmatrix} 1 & \theta_e & \theta_e \\ \theta_e & 1 & \theta_e^2 \\ \theta_e & \theta_e^2 & 1 \end{pmatrix}\right).$$

To implement estimators $\hat{\beta}_1$, $\hat{\beta}_{1,M}$ and $\hat{\beta}_{1,G}$, I use 3rd order B splines for my estimator $\hat{m}(W)$ with knot sequences chosen according to assumption 1, 6th order gaussian kernels and rule of thumb bandwidths for Nadaraya Watson estimator $\hat{m}_G(W)$, I use 2nd order gaussian kernels and rule of thumb bandwidths for all required density estimators in step 2, and I use 6th order gaussian kernels and cross validated bandwidths for all Nadaraya Watson type estimators in step 3. Data for this study was generated using 500 iterations for each of the 12 unique combinations of $(\text{DGP}, \theta_e, n)$. The results of this Monte Carlo exercise are summarized in Table 1, which lists the bias (B), the standard deviation (S), the root mean squared error (R), and the median squared error (D), for each estimator. Also listed are the mean squared error (M1), and (M2) for the estimation of $m_1(W)$ and $m_2(W)$ respectively. Note, that the Table and Figures summarize only those estimated values within the 2% - 98% sample quantile range. Also note that M1 and M2 values for $\hat{\beta}_{1,M}$ are left blank as in this case $m_1(W)$ and $m_2(W)$ are taken to be known functions. In addition to the results summarized in Table 1, Rosenblatt kernel density estimates for the absolute error of all three estimators are shown in Figure 1 for a sample size of $n = 200$, and in Figure 2 for a sample size of $n = 400$.

The additive model presented in the paper is a restriction on the model developed in Geng et al. (2016) who assume no additivity. As a result one should expect that the estimator in this paper will have better finite sample performance for two reasons. Firstly, since both primary equations in this study are additive functions, the estimator in this paper should perform better than Geng et al. (2016) regardless of the degree of endogeneity or sample size as it is has a sharper specification. Secondly, as discussed in the introduction, using an additive B spline estimator for $m(W)$ in this setting avoids the "curse of dimensionality" relative to estimators that do not exploit additivity. Consequently, the estimators for $m_1(W)$ and $m_2(W)$ in this paper should outperform Geng et al. (2016) who use Nadaraya Watson estimators. Table 1 demonstrates that this is true, as in all cases the mean squared error for $\hat{m}(W)$ is smaller than $\hat{m}_G(W)$.

The structure of this Monte Carlo exercise is designed to demonstrate changes in the finite sample properties of each estimator due to variations in data generating process, sample size, and degree of endogeneity. As for variations in sample size, in Table 1 it is clear that increasing the sample size has the expected effect since, in all cases, the bias, standard deviation, and mean squared error of each estimator decline, supporting the consistency results derived for each. As the level of endogeneity increases one should expect that the two disadvantages inherent in Geng et al. (2016) discussed above should be compounded since comparatively poorer estimates $\hat{m}_G(W)$, due to a more general specification for $m(W)$, are used in a second stage estimator $\hat{\beta}_{1,G}$ which is derived using another more general specification. In Table 1 and both Figures, one can see that this is the case as the negative effect that increasing levels of endogeneity have on all estimates is disproportionately large on $\hat{\beta}_{1,G}$.

Table 1: Finite Sample Performance

	$\theta_e = 0.3$						$\theta_e = 0.6$						$\theta_e = 0.9$					
	B	S	R	D	M1	M2	B	S	R	D	M1	M2	B	S	R	D	M1	M2
DGP=1	n=200																	
$\hat{\beta}_1$	-0.5982	0.1075	0.3694	0.3618	0.2938	0.3066	-0.6400	0.1016	0.4199	0.4104	0.2998	0.3102	-0.6704	0.0967	0.4587	0.4516	0.2980	0.3089
$\hat{\beta}_{1,G}$	-0.6641	0.1487	0.4631	0.4322	0.7041	0.6926	-0.8065	0.1362	0.6690	0.6465	0.7046	0.6905	-0.8787	0.1112	0.7844	0.7747	0.6997	0.6964
$\hat{\beta}_{1,M}$	-0.6177	0.1080	0.3932	0.3846			-0.6440	0.1002	0.4248	0.4130			-0.6609	0.0936	0.4455	0.4360		
	n=400																	
$\hat{\beta}_1$	-0.5746	0.0747	0.3358	0.3330	0.2031	0.2281	-0.6154	0.0729	0.3840	0.3771	0.2057	0.2252	-0.6379	0.0650	0.4111	0.4098	0.2058	0.2269
$\hat{\beta}_{1,G}$	-0.6551	0.0998	0.4391	0.4274	0.6885	0.6892	-0.7884	0.0888	0.6294	0.6202	0.6878	0.6850	-0.8517	0.0779	0.7315	0.7260	0.6880	0.6876
$\hat{\beta}_{1,M}$	-0.5856	0.0757	0.3486	0.3445			-0.6223	0.0736	0.3927	0.3790			-0.6396	0.0653	0.4134	0.4089		
DGP=2	n=200																	
$\hat{\beta}_1$	-0.1449	0.1217	0.0358	0.0223	0.2967	0.3094	-0.1879	0.1116	0.0477	0.0353	0.2989	0.3144	-0.1927	0.1026	0.0476	0.0385	0.2945	0.3128
$\hat{\beta}_{1,G}$	-0.2548	0.1542	0.0886	0.0657	0.7018	0.6917	-0.3964	0.1553	0.1812	0.1648	0.7002	0.6937	-0.4939	0.1369	0.2626	0.2497	0.7057	0.7007
$\hat{\beta}_{1,M}$	-0.1388	0.1255	0.0350	0.0202			-0.1587	0.0850	0.0324	0.0235			-0.1587	0.0850	0.0324	0.0235		
	n=400																	
$\hat{\beta}_1$	-0.1321	0.0794	0.0237	0.0172	0.2070	0.2255	-0.1708	0.0749	0.0348	0.0288	0.2047	0.2253	-0.1841	0.0701	0.0388	0.0325	0.2067	0.2304
$\hat{\beta}_{1,G}$	-0.2204	0.1063	0.0598	0.0514	0.6866	0.6866	-0.3548	0.1083	0.1376	0.1296	0.6855	0.6821	-0.4696	0.0988	0.2303	0.2265	0.6897	0.6839
$\hat{\beta}_{1,M}$	-0.1255	0.0773	0.0217	0.0148			-0.1621	0.0700	0.0312	0.0260			-0.1609	0.0583	0.0293	0.0249		

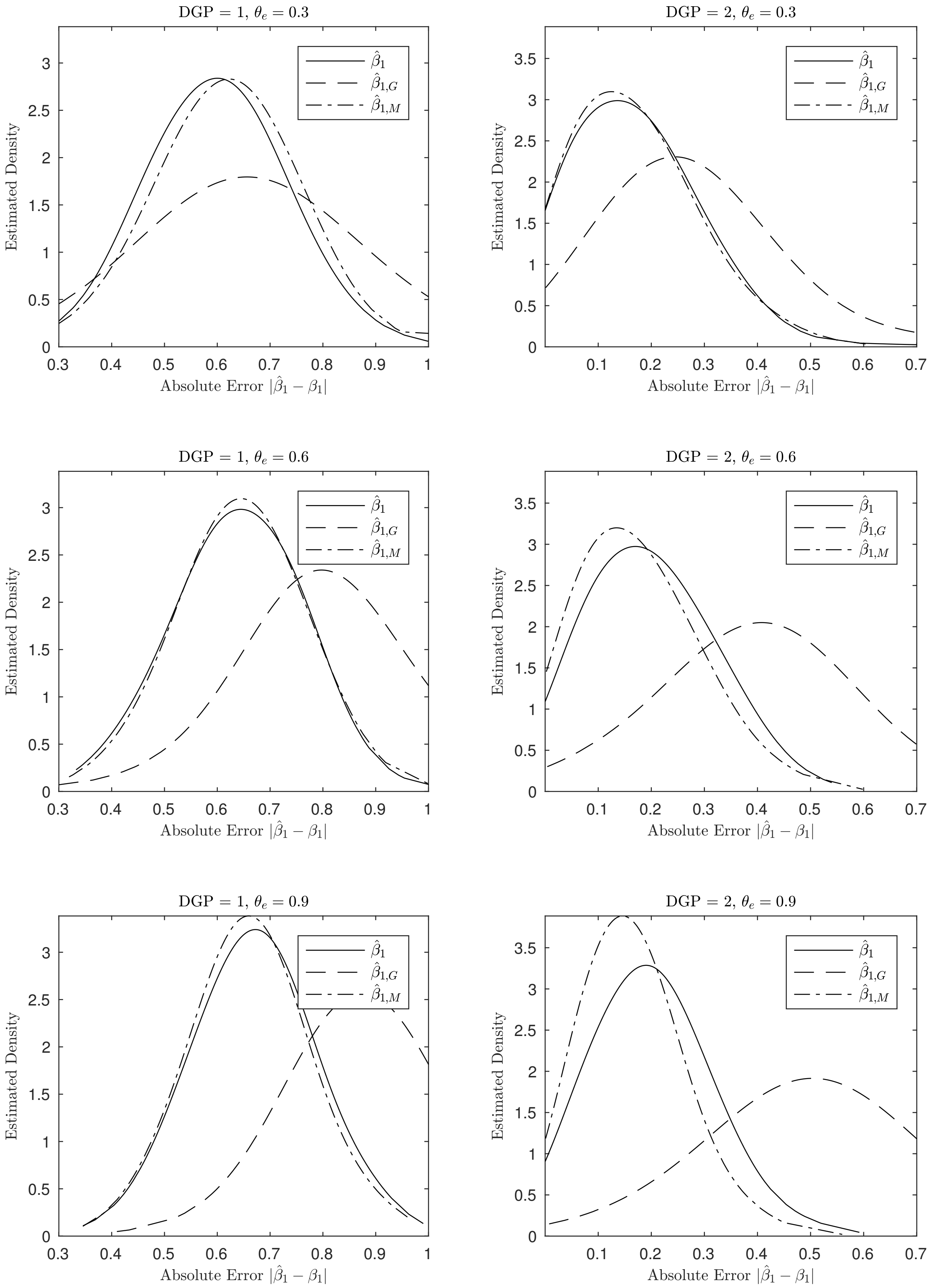

Figure 1: Estimated Densities for Absolute Error of estimators of β_1, $n = 200$

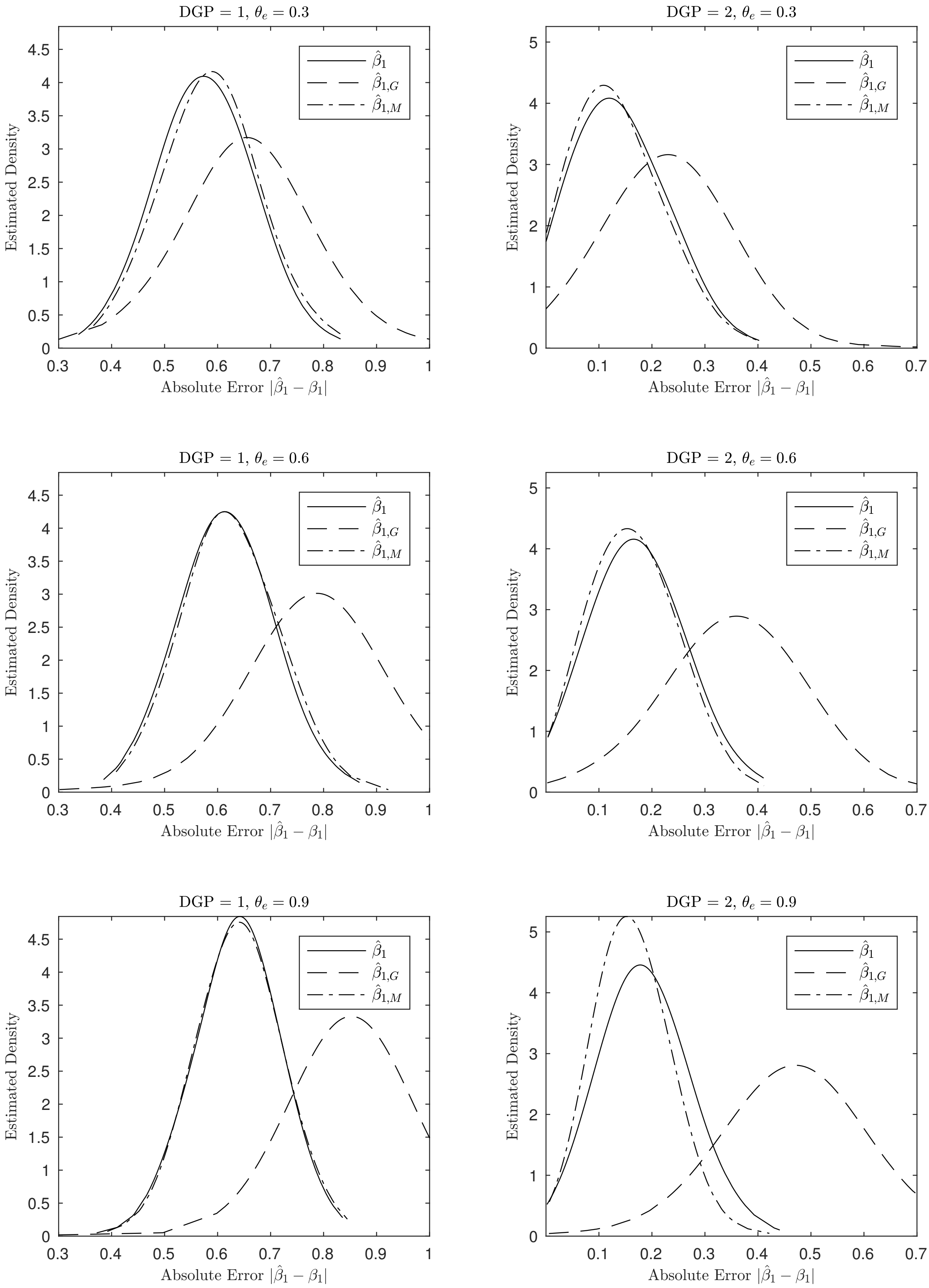

Figure 2: Estimated Densities for Absolute Error of estimators of β_1, $n = 400$

Recall that the sole difference between estimators $\hat{\beta}_1$ and $\hat{\beta}_{1,M}$ is whether $m(W)$ is a known or estimated. In Theorem 3, I have shown that the asymptotic variance of $\hat{\beta}_1$ is identical to $\hat{\beta}_{1,M}$, a property which I call Oracle efficiency in the somewhat atypical sense that I describe in the introduction. The purpose of comparing $\hat{\beta}_1$ and $\hat{\beta}_{1,M}$ here is to determine to what extent this Oracle property is evident in finite samples. The findings presented in Table 1, and both Figures reinforce the Oracle efficiency result derived in Theorem 3 as one can clearly see that the density of the absolute error of $\hat{\beta}_1$, and $\hat{\beta}_{1,M}$ are practically coincident in 9 of the 12 panels and the tabulated results are likewise very similar. This is in some ways surprising as one would not expect the asymptotic Oracle efficiency of $\hat{\beta}_1$ to so clearly manifest itself in finite samples. These results show that not only is there asymptotically no penalty to estimating $m(W)$ but there is also very little penalty in estimating $m(W)$ in finite samples.

5 Summary and Conclusion

In this paper I have provided an asymptotic characterization of a estimator for the finite dimensional parameter of the partially linear primary equation in a triangular system of equations constructed to handle nonparametrically defined endogenous regressors using the control function approach. Theorem 3 shows that this estimator is consistent, $\sqrt{n}$ asymptotically normal, and Oracle efficient. Additionally, both Theorem 3 and the Monte Carlo exercise presented in the previous section show that in an additive context, the estimator presented in this paper practically identical to the Oracle estimator of Manzan and Zerom (2005) and is superior to the estimator developed in Geng et al. (2016). One extension requiring further study is the asymptotic characterization of a nonparametric estimation procedure for $h(X)$.

6 Appendix

Adopt the following notation $\mathsf{S}_1^d(Y_i) \equiv H_1^d(Y_i) - \hat{H}_1^d(Y_i)$, $\mathsf{S}_2^d(Y_i) \equiv H_2^d(Y_i) - \hat{H}_2^d(Y_i)$, $\mathsf{S}_1^d(Z_i) \equiv H_1^d(Z_i) - \hat{H}_1^d(Z_i)$, $\mathsf{S}_2^d(Y_i) \equiv H_2^d(Z_i) - \hat{H}_2^d(Z_i)$, $K_{1ji}(X_d) \equiv K_1[b_1^{-1}(X_{dj} - X_{di})]$, and $K_{2ji}(V_d) \equiv K_2[b_1^{-1}(V_{dj} - V_{di})]$ furthermore define,

$$\mathsf{C}_{2ji}^d = (\hat{\theta}_{2j}^d - \theta_{2j}^d)Z_{cj} + \theta_{2j}^d(Z_{cj} - E[Z_{cj}|X_j, V_j]) + \theta_{2j}^d(E[Z_{cj}|X_j, V_j] - H_2^d(Z_{cj})) + \theta_{2j}^d(H_2^d(Z_{cj}) - H_2^d(Z_{ci})),$$
$$\mathsf{C}_{1ji}^d = (\hat{\theta}_{1j}^d - \theta_{1j}^d)Z_{cj} + \theta_{1j}^d(Z_{cj} - E[Z_{cj}|X_j, V_j]) + \theta_{1j}^d(E[Z_{cj}|X_j, V_j] - H_1^d(Z_{cj})) + \theta_{1j}^d(H_1^d(Z_{cj}) - H_1^d(Z_{ci})),$$
$$\mathsf{C}_{2ji}^{*d} = (\hat{\theta}_{2j}^d - \theta_{2j}^d)Z_{cj} + \theta_{2j}^d(Z_{cj} - E[Z_{cj}|X_j, V_j]) + \theta_{2j}^d(E[Z_{cj}|X_j, V_j] - H_2^d(Z_{cj})) + \theta_{2j}^d H_2^d(Z_{cj}),$$
$$\mathsf{C}_{1ji}^{*d} = (\hat{\theta}_{1j}^d - \theta_{1j}^d)Z_{cj} + \theta_{1j}^d(Z_{cj} - E[Z_{cj}|X_j, V_j]) + \theta_{1j}^d(E[Z_{cj}|X_j, V_j] - H_1^d(Z_{cj})) + \theta_{1j}^d H_1^d(Z_{cj}).$$

Furthermore,

$$K_{2ji}^{(m)}(V_d) \equiv K_2^{(m)}[b_2^{-1}(V_{dj} - V_{di})] = \frac{d^m}{d^m\gamma}K_2(\gamma)\Big|_{\gamma=b_2^{-1}(V_{dj}-V_{di})},$$
$$K_{2ji}^{(4)}(\tilde{V}_d) \equiv \frac{d^m}{d^m\gamma}K_2(\gamma)\Big|_{\gamma=\lambda b_2^{-1}(\hat{V}_{dj}-\hat{V}_{di}))+(1-\lambda)b_2^{-1}(V_{dj}-V_{di}))},$$
$$D_dK_{3ji} \equiv D_dK_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] = \frac{\partial}{\partial\gamma_{2d}}K_3[(\gamma_1', \gamma_2')']\Big|_{(\gamma_1',\gamma_2')'=H^{-1}\{(X_j',V_j')'-(X_i',V_i')'\}},$$
$$D_{kd}K_{3ji} \equiv D_{kd}K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] = \frac{\partial^2}{\partial\gamma_{2d}\partial\gamma_{2k}}K_3[(\gamma_1', \gamma_2')']\Big|_{(\gamma_1',\gamma_2')'=H^{-1}\{(X_j',V_j')'-(X_i',V_i')'\}},$$
$$D_{mkd}K_{3ji} \equiv D_{mkd}K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] = \frac{\partial^3}{\partial\gamma_{2d}\partial\gamma_{2k}\partial\gamma_{2m}}K_3[(\gamma_1', \gamma_2')']\Big|_{(\gamma_1',\gamma_2')'=H^{-1}\{(X_j',V_j')'-(X_i',V_i')'\}\}},$$
$$D_{qmkd}K_{3ji} \equiv D_{mkd}K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] = \frac{\partial^4}{\partial\gamma_{2d}\partial\gamma_{2k}\partial\gamma_{2m}\partial\gamma_{2q}}K_3[(\gamma_1', \gamma_2')']\Big|_{(\gamma_1',\gamma_2')'=H^{-1}\{(X_j',V_j')'-(X_i',V_i')'\}}.$$

Furthermore,

$$\mathsf{S}_{1n}^d(Y) = \left[\mathsf{S}_1^d(Y_1)\ \ \mathsf{S}_1^d(Y_2)\ \ \cdots\ \ \mathsf{S}_1^d(Y_n)\right]', \qquad \mathsf{S}_{2n}^d(Y) = \left[\mathsf{S}_2^d(Y_1)\ \ \mathsf{S}_2^d(Y_2)\ \ \cdots\ \ \mathsf{S}_2^d(Y_n)\ \right]',$$
$$\mathsf{S}_{1n}^d(Z) = \left[\mathsf{S}_1^d(Z_1)\ \ \mathsf{S}_1^d(Z_2)\ \ \cdots\ \ \mathsf{S}_1^d(Z_n)\right]', \qquad \mathsf{S}_{2n}^d(Z) = \left[\mathsf{S}_2^d(Z_1)\ \ \mathsf{S}_2^d(Z_2)\ \ \cdots\ \ \mathsf{S}_2^d(Z_n)\ \right]',$$
$$\mathbf{K}_{1i}(X_d) = \left[K_{11i}(X_d)\ \ K_{12i}(X_d)\ \ \cdots\ \ K_{1ni}(X_d)\right]', \qquad \mathbf{K}_{2i}^{(m)}(V_d) = \left[K_{21i}^{(m)}(V_d)\ \ K_{22i}^{(m)}(V_d)\ \ \cdots\ \ K_{2ni}^{(m)}(V_d)\right]',$$
$$\mathbf{D}_dK_{3i} = \left[D_dK_{31i}\ \ D_dK_{32i}\ \ \cdots\ \ D_dK_{3ni}\right]', \qquad \mathbf{M}_d = [m_d(W_1)\ \ m_d(W_2)\ \ \cdots\ \ m_d(W_n)]',$$
$$\mathbf{M}_d^{l_n} = [m_d^{l_n}(W_1)\ \ m_d^{l_n}(W_2)\ \ \cdots\ \ m_d^{l_n}(W_n)]', \qquad \hat{\mathbf{M}}_d^{l_n} = [\hat{m}_d^{l_n}(W_1)\ \ \hat{m}_d^{l_n}(W_2)\ \ \cdots\ \ \hat{m}_d^{l_n}(W_n)]',$$

and $I(-j)$ is an identity matrix of appropriate dimension with the jth diagonal entry set to zero. Also,

$$\dot{\boldsymbol{\zeta}}_c = diag(\{\zeta_{ci}\}_{i=1}^n), \qquad \dot{\boldsymbol{\Theta}}_{1n}^d = diag\big(\{\theta_{1i}^d\}_{i=1}^n\big), \qquad \dot{\boldsymbol{\Theta}}_{2n}^d = diag\big(\{\theta_{2i}^d\}_{i=1}^n\big),$$
$$\dot{\mathbf{p}}(X_d)^{-1} = diag\big(\{p(X_{di})^{-1}\}_{i=1}^n\big), \qquad \dot{\mathbf{p}}(V_d)^{-1} = diag\big(\{p(V_{di})^{-1}\}_{i=1}^n\big), \qquad \dot{\mathbf{u}}_n = diag\Big(\{u_i\}_{i=1}^n\Big),$$
$$\dot{\boldsymbol{\eta}}_{2c}^d = diag(\{\eta_{2cj}^d\}_{j=1}^n) \qquad \dot{\boldsymbol{\eta}}_{1c}^d = diag(\{\eta_{1cj}^d\}_{j=1}^n) \qquad \dot{\boldsymbol{\rho}}_c = diag(\{\rho_{ci}\}_{i=1}^n)$$
$$\dot{\mathbf{H}}_2^d(Z_c) = diag(\{H_2^d(Z_{ci})\}_{i=1}^n) \qquad \dot{\mathbf{V}}_d = diag(\{V_{di}\}_{i=1}^n) \qquad \dot{\mathbf{V}}_{dj} = V_{dj}I_{l_n}$$

Also conditioning on S_i indicates conditioning on everything indexed by i and conditioning on S_{-i} indicates conditioning on everything not indexed by i.

Proof of Lemma 1

Part i) By Assumption A5

$$E\left[\phi\left(\sum_{j=1}^D h_j(X_j) + \sum_{j=1}^D f_j(V_j)\right)\Bigg| X_d\right] = \sum_{d=1}^D E(\phi h_d(X_d)|X_d) + \sum_{j=1}^D E(\phi f_d(V_d)|X_d)$$

$$= h_d(X_d) \int \frac{g(X,V)}{p(X,V)} \frac{p(X,V)}{p(X_d)} d(X_{-d},V) + \int \left(\sum_{j \neq d}^{D} h_j(X_j) + \sum_{j=1}^{D} f_j(V_j) \right) \frac{g(X,V)}{p(X,V)} \frac{p(X,V)}{p(X_d)} d(X_{-d},V)$$

$$= h_d(X_d) \int g(X_{-d},V) d(X_{-d},V) + \sum_{j \neq d}^{D} \int h_j(X_j) g(X_{-d},V) d(X_{-d},V) + \sum_{j=1}^{D} \int f_j(V_j) g(X_{-d},V) d(X_{-d},V)$$

$$= h_d(X_d) + \sum_{j \neq d}^{D} E[h_j(X_j)] + \sum_{j=1}^{D} E[f_j(V_j)] = h_d(X_d).$$

In a similar manner, $E\left[\phi \left(\sum_{j=1}^{D} h_j(X_j) + \sum_{j=1}^{D} f_j(V_j) \right) \middle| V_d \right] = f_d(V_d)$. Additionally note that,

$$\begin{aligned}
E[\phi(Y - Z'\beta_1)] &= \mu_Y - \mu_Z'\beta_1 \\
&= \beta_0 E[\phi] + \sum_{d=1}^{D} E[\phi \, h_d(X_j)] + \sum_{d=1}^{D} E[\phi f_d(V_j)] + E[\phi\varepsilon] \\
&= \beta_0 \int \frac{g(X,V)}{p(X,V)} p(X,V) dX dV + \sum_{d=1}^{D} \int h_d(X_d) \frac{g(X,V)}{p(X,V)} p(X,V) dX dV \\
&\qquad + \sum_{d=1}^{D} \int f_d(V_d) \frac{g(X,V)}{p(X,V)} p(X,V) dX dV + E[\phi E(\varepsilon | Z, X, V)] \\
&= \beta_0 + \sum_{d=1}^{D} E[h_d(X_d)] + \sum_{d=1}^{D} E[f_d(V_d)] \\
&= \beta_0.
\end{aligned}$$

As a result, provided μ_Y and μ_Z exist, β_0 is identified whenever β_1 is. Next consider,

$$\begin{aligned}
H_1^d(Y) &= \beta_0 E[\phi | X_d] + E[\phi Z | X_d]'\beta_1 + E\left[\phi \left(\sum_{j=1}^{D} h_j(X_j) + \sum_{j=1}^{D} f_j(V_j) \right) \middle| X_d \right] + E[\phi E(\varepsilon | Z, X, V) | X_d] \\
&= \beta_0 + H_1^d(Z)'\beta_1 + h_d(X_d).
\end{aligned}$$

and,

$$\begin{aligned}
H_2^d(Y) &= \beta_0 E[\phi | V_d] + E[\phi Z | V_d]'\beta_1 + E\left[\phi \left(\sum_{j=1}^{D} h_j(X_j) + \sum_{j=1}^{D} f_j(V_j) \right) \middle| V_d \right] + E[\phi E(\varepsilon | Z, X, V) | V_d] \\
&= \beta_0 + H_2^d(Z)'\beta_1 + f_d(V_d).
\end{aligned}$$

Combining these statements gives,

$$\begin{aligned}
H(Y) &= \sum_{d=1}^{D} \left[H_1^d(Y) + H_2^d(Y) \right], \\
&= 2D\beta_0 + \sum_{d=1}^{D} \left[H_1^d(Z)'\beta_1 + H_2^d(Z)' \right] \beta_1 + \sum_{d=1}^{D} [h_d(X_d) + f_d(V_d)] \\
&= 2D\beta_0 + H(Z)'\beta_1 + h(X) + f(V).
\end{aligned}$$

Next, recalling that $\beta_0 = \mu_Y - \mu_Z'\beta_1$,

$$\begin{aligned}
H^*(Y) &= H(Y) - (2D-1)\mu_Y \\
&= 2D\beta_0 + H(Z) + h(X) + f(V) - (2D-1)\mu_Y
\end{aligned}$$

$$= \beta_0 + (2D-1)\mu_Y - (2D-1)\mu_Z'\beta_1 + H(Z)'\beta_1 + h(X) + f(V) - (2D-1)\mu_Y$$
$$= \beta_0 + H^*(Z)'\beta_1 + h(X) + f(V).$$

Now, subtracting the above from equation (6) gives,

$$Y - H^*(Y) = (Z - H^*(Z))'\beta_1 + \varepsilon.$$

Lastly, premultiplying both sides by $Z - H^*(Z)$, and taking an expectation gives,

$$E\Big[(Z - H^*(Z))(Y - H^*(Y))\Big]$$
$$= E\Big[(Z - H^*(Z))(Z - H^*(Z))'\Big]\beta_1 + E\Big[(Z - H^*(Z))E[\varepsilon|Z, X, V]\Big]$$
$$= E\Big[(Z - H^*(Z))(Z - H^*(Z))'\Big]\beta_1.$$

Hence, if $E\Big[(Z - H^*(Z))(Z - H^*(Z))'\Big]$ is positive definite, then β_1 is identified. Now by A3(i) $Z \in L^2(\Omega, \mathcal{A}, P)$ so that by that projection theorem $Z - E[Z|X, V]$ is orthogonal to the space of square integrable functions of X, V. In particular $E[Z|X, V] - H^*(Z)$, as a result,

$$E\Big[(Z - H^*(Z))(Z - H^*(Z))'\Big]$$
$$= E\left[(Z - E[Z|X, V] + E[Z|X, V] - H^*(Z))(Z - E[Z|X, V] + E[Z|X, V] - H^*(Z))'\right]$$
$$= E\left[(Z - E[Z|X, V])(Z - E[Z|X, V])'\right] + E\left[(E[Z|X, V] - H^*(Z))(E[Z|X, V] - H^*(Z))'\right]$$
$$= E\left[\rho\rho'\right] + E\left[\eta\eta'\right].$$

Consequently β_1 is identified if either, $E[\rho\rho']$ or $E[\eta\eta']$ is positive definite. **Part ii)**Recall that,

$$Y - H^*(Y) = (Z - H^*(Z))'\beta_1 + \varepsilon$$

and note that for any measurable function $L(X, V) : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$, $E\left(L(X, V)\varepsilon\right) = E\left(L(X, V)E[\varepsilon|Z, X, V]\right) = 0$. As a result, if we let $L(X, V) = \sqrt{\phi(X, V)}$, $\sqrt{\phi(X, V)}(Y - H^*(Y)) = \sqrt{\phi(X, V)}(Z - H^*(Z))'\beta_1 + \sqrt{\phi(X, V)}\varepsilon$. Then, premultiplying by $(Z - H^*(Z))\sqrt{\phi(X, V)}$ and taking expectations,

$$E\Big[(Z - H^*(Z))\phi(X, V)(Y - H^*(Y))\Big]$$
$$= E\Big[(Z - H^*(Z))\phi(X, V)(Z - H^*(Z))'\Big]\beta_1 + E\Big[(Z - H^*(Z))\phi(X, V)E[\varepsilon|Z, X, V]\Big]$$
$$= E\Big[(Z - H^*(Z))\phi(X, V)(Z - H^*(Z))'\Big]\beta_1$$
$$= E[\zeta\phi\zeta']\beta_1.$$

Consequently if $E[\zeta\phi\zeta']$ is positive definite then β_1 is identified.

□

Lemma 2. *$E[\phi(Z_c - H^*(Z_c))|X_d] = 0$, $E[\phi(Z_c - H^*(Z_c))|V_d] = 0$, $E[\phi(Y - H^*(Y))|X_d] = 0$, and $E[\phi(Y - H^*(Y))|V_d] = 0$.*

Proof. Let A be any of the real valued random variables in $[\ Y\ Z_1\ Z_2\ \ldots\ Z_p]'$, and recall from Lemma 1 that $E[\phi|X_d] = 1$ and $E[\phi|V_d] = 1$.

$$E[\phi(A - H^*(A))|X_d]$$
$$= E[\phi A|X_d] - E\left[\phi\left(\sum_{j=1}^{D} H_1^j(A) + \sum_{j=1}^{D} H_2^j(A)\right)\middle| X_d\right] + (2D-1)E[\phi A]E[\phi|X_d]$$
$$= H_1^d(A) - H_1^d(A)E[\phi|X_d] - \sum_{j\neq d}^{D} \int \frac{g(X_{-d}, V)}{p(X, V)} E[\phi A|X_j] p(X, V) dX dV$$

$$
-\sum_{j=1}^{D}\int\frac{g(X_{-d},V)}{p(X,V)}E[\phi A|V_j]p(X,V)dXdV+(2D-1)E[\phi A]E[\phi|X_d]
$$

$$
=H_1^d(A)-H_1^d(A)-\sum_{j\neq d}^{D}\int E[\phi A|X_j]p(X_j)dX_j-\sum_{j=1}^{D}\int E[\phi A|V_j]p(V_j)dV_j+(2D-1)E[\phi A]
$$

$$
=(2D-1)E[\phi A]-\sum_{j\neq d}^{D}E[\phi A]-\sum_{j=1}^{D}E[\phi A]=0.
$$

Similarly, $E[\phi(A-H^*(A))|V_d]=0$. □

Lemma 3. *Under Assumption A2*

i) $l_n^{-k}=o(n^{-1/2})$,

ii) $l_n/[\sqrt{n}b_2]=o(n^{-1/4})$,

iii) $l_n/[n(h_1^D h_2^{D+2})^{1/2}]=o(n^{-1/2})$,

iv) $l_n^{-k}/b_2=o(n^{-1/4})$,

v) $l_n/[nb_2^{3/2}]=o(n^{-1/2})$,

vi) $L_n/b_2=o(n^{-1/4})$,

vii) $L_n^4/b_2^5=o(n^{-1/2})$,

viii) $\log(n)^{1/2}L_n/b_2^{3/2}=o(1)$,

ix) $L_n/b_2^2=o(1)$,

x) $\log(n)/[(n-1)b_2^3]=o(n^{-1/4})$,

xi) $L_n/h_2=o(n^{-1/4})$,

xii) $L_n^4/h_2^8=o(n^{-1/2})$,

xiii) $h_0^{\nu_0}=o(n^{-1/4})$,

xiv) $\log(n)/[nh_o]=o(n^{-1/2})$,

xv) $M_{1n}=o(n^{-1/4})$,

xvi) $h_1^{\nu_3}+h_2^{\nu_3}=o(n^{-1/4})$,

xvii) $\log(n)/[nh_1^D h_2^{D+2}]=o(n^{-1/2})$,

xviii) $M_{2n}=o(n^{-1/4})$,

xix) $l_n/[nh_1^D h_2^{D+2}]=o(1)$,

xx) $b_1^{\nu_1}=o(n^{-1/4})$,

xxi) $\log(n)/[nb_1]=o(n^{-1/2})$,

xxii) $N_{1n}=o(n^{-/4})$,

xxiii) $b_2^{\nu_2}=o(n^{-1/4})$,

xxiv) $\log(n)/[nb_2]=o(n^{-1/2})$,

xxv) $N_{2n}=o(n^{-1/4})$,

xxvi) $M_n,\mathcal{L}_{0n}=o(n^{-1/4})$,

xxvii) $\mathcal{L}_{1n},\mathcal{L}_{2n},\mathcal{L}_n=o(n^{-1/4})$.

Proof. Note, in the following let $\varepsilon>0$ be an arbitrarily small real number, and for a sequence of positive real numbers $\{a_n\}_{n=1}^{\infty}$ and constants $z,C\in\mathbb{R}^+$, $a_n\sim n^z$ means that $0<a_n/n^z\leq C<\infty$ for all $n\in\mathbb{N}$.

Part *i*) : By A6: $a>1/2k$, let $a=1/2k+\varepsilon$,

$$
l_n^{-k}n^{1/2}\sim n^{-ak}n^{1/2}=n^{-k(1/2k+\varepsilon)+1/2}=n^{-k\varepsilon}=o(1).
$$

Part *ii*) : By A6: $b<1/4-a$, let $b=1/4-a-\varepsilon$,

$$
l_n(n^{1/2}b_2)^{-1}n^{1/4}\sim n^a n^{-1/2}n^{1/4-a-\varepsilon}n^{1/4}=n^{-\varepsilon}=o(1).
$$

Part *iii*) : By A6: $c<1/4(D+1)$ let $c=1/4(D+1)-\varepsilon$,

$$
\begin{aligned}
\left(l_n/[n(h_1^D h_2^{D+2})^{1/2}]\right)n^{1/2}&=\left(l_n/\sqrt{n}\right)\left([nh_1^D h_2^{D+2}]^{-1/2}\right)n^{1/2}\\
&\sim o(1)n^{-1/4}n^{-1/2+(D+1)(1/4(D+1)-\varepsilon)}n^{1/2}\\
&=o(1)n^{-3/4+1/4+1/2-(D+1)\varepsilon}=o(1)n^{-(D+1)\varepsilon}=o(1).
\end{aligned}
$$

Part *iv*) : By A6: $a>1/2k$, $b<1/4-a$, let $b=1/4-a-\varepsilon$ and $a=1/2k+\varepsilon$,

$$
l_n^{-k}b_2^{-1}n^{1/4}\sim n^{-ak}n^{1/4-a-\varepsilon}n^{1/4}=n^{-k(1/2k+\varepsilon)}n^{1/2-a-\varepsilon}=n^{-a-(1+k)\varepsilon}=o(1).
$$

Part *v*) : By A6: $a > 1/2k$, $b < 1/4 - a$, let $b = 1/4 - a - \varepsilon$ and $a = 1/2k + \varepsilon$,

$$\begin{aligned} l_n(nb_2^{3/2})^{-1}n^{1/2} &= (l_n/n^{1/2})(n^{1/2}b_2^{3/2})^{-1}n^{1/2} = o(n^{-1/4})b_2^{3/2} \sim o(1)n^{-1/4}n^{1/4-a-\varepsilon} \\ &= o(1)n^{-a-\varepsilon} = o(1). \end{aligned}$$

Note, the second equality is due to (ii).

Part *vi*) : By A6: $a > 1/2k$, $b < 1/4 - a$, let $b = 1/4 - a - \varepsilon$ and $a = 1/2k + \varepsilon$,

$$\begin{aligned} L_n b_2^{-1} n^{1/4} &= l_n n^{-1/2} b_2^{-1} n^{1/4} + l_n^{1/2-k} b_2^{-1} n^{1/4} \\ &\sim n^a n^{-1/4} n^b + l_n^{-k} n^{a/2} n^b n^{1/4} \\ &= n^{a-1/4+1/4-a-\varepsilon} + o(n^{-1/2}) n^{a/2+1/4-a-\varepsilon+1/4} \\ &= n^{-\varepsilon} + o(1) n^{-1/2} n^{1/2-a/2-\varepsilon} \\ &= n^{-\varepsilon} + o(1) n^{-a/2-\varepsilon} = o(1). \end{aligned}$$

Note, the third equality is due to (i).

Part *vii*) : By A6: $a > 1/2k$, $b < 1/4 - a$, let $a = 1/2k + \varepsilon$ and $b = 1/4 - 1/2k - \varepsilon$,

$$L_n^4 b_2^{-5} n^{1/2} = (L_n/b_2)^4 b_2^{-1} n^{1/2} \sim o(n^{-1}) n^b n^{1/2} = o(1) n^{-1/2} n^{1/4-1/2k-\varepsilon} = n^{-(1/4+1/2k+\varepsilon)} = o(1).$$

Part *viii*) : By A6: $a > 1/2k$, $b < 1/4 - a$, let $a = 1/2k + \varepsilon$ and $b = 1/4 - 1/2k - \varepsilon$,

$$\begin{aligned} \log(n)^{1/2} L_n b_2^{-3/2} &= \log(n)^{1/2} \big(L_n b_2^{-1}\big) b_2^{-1/2} \sim o(n^{-1/4}) \log(n)^{1/2} n^{b/2} \\ &= o(1) \log(n)^{1/2} n^{-1/4+1/8-1/4k-\varepsilon/2} \\ &= o(1) \log(n)^{1/2} n^{-1/8-1/4k-\varepsilon} = o(1), \end{aligned}$$

by L'Hôpital's Rule.

Part *ix*) : By A6: $a > 1/2k$, $b < 1/4 - a$, let $a = 1/2k + \varepsilon$ and $b = 1/4 - 1/2k - \varepsilon$,

$$L_n b_2^{-2} = (L_n b_2^{-1}) b_2^{-1} = o(n^{-1/4}) n^b = o(1) n^{-1/4} n^{1/4-1/2k-\varepsilon} = o(1) n^{-1/2k-\varepsilon} = o(1).$$

Note, the second equality is due to (vi).

Part *x*) : By A6: $a > 1/2k$, $b < 1/4 - a$, let $a = 1/2k + \varepsilon$ and $b = 1/4 - 1/2k - \varepsilon$,

$$\log(n)[(n-1)b_2^3]^{-1} n^{1/4} \sim \log(n) n^{-3/4} n^{3b} = \log(n) n^{-3/4+3/4-3/4k-3\varepsilon} = \log(n) n^{-3/2k-3\varepsilon} = o(1),$$

by L'Hôpital's Rule and since for $n \geq 2$, $1/2 \leq (n-1)/n \leq 1$.

Part *xi*) : By A6: $a > 1/2k$, $b < 1/4 - a$, and $c < (2k-1)/4k(D+1)$, let $a = 1/2k + \varepsilon$, $b = 1/4 - 1/2k - \varepsilon = (2k-1)/8k - \varepsilon$ and $c = (2k-1)/4k(D+1) - \varepsilon \leq (2k-1)/8k - \varepsilon$,

$$\big(L_n/h_2\big) n^{1/4} = (L_n b_2^{-1}) b_2 h_2^{-1} n^{1/4} \sim o(n^{-1/4}) n^{-b} n^c n^{1/4} \leq o(1) n^{-(2k-1)/8k+\varepsilon+(2k-1)/8k-\varepsilon} = o(1).$$

Part *xii*) : By A6: $c < 1/4(D+1)$ let $c = 1/4(D+1) - \varepsilon$

$$\big(L_n^4/h_2^8\big) n^{1/2} = \big(L_n/h_2\big)^4 h_2^{-4} n^{1/2} \sim o(n^{-1}) n^{4c} n^{1/2} = o(1) n^{-1/2+4/4(D+1)-\varepsilon} < o(1) n^{-\varepsilon} = o(1).$$

Part *xiii*) : By A6: $\nu_o > 1/4f$ let $\nu_0 = 1/4f + \varepsilon$,

$$h_0^{\nu_0} n^{1/4} \sim n^{-f(1/4f+\varepsilon)} n^{1/4} = n^{-1/4-\varepsilon f+1/4} = n^{-\varepsilon f} = o(1).$$

Part *xiv*) : By A6: $f < 1/2$ let $f = 1/2 - \varepsilon$,

$$\log(n) n^{-1} h_o^{-1} n^{1/2} \sim \log(n) n^{-1/2} n^f = \log(n) n^{-1/2} n^{1/2-\varepsilon} = \log(n) n^{-\varepsilon} = o(1).$$

By L'Hôpital's Rule.

Part *xv*) : By *xii*) and *xiii*), $M_{1n} = \left[\frac{\log(n)}{nh_0}\right]^{1/2} + h_0^{\nu_0} = o(n^{-1/4}) + o(n^{-1/4})$.

Part *xvi*) : By A6: $\nu_3 > 1/4c$ let $\nu_3 = 1/4c + \varepsilon$,

$$h_1^{\nu_3} n^{1/4} + h_2^{\nu_3} n^{1/4} \sim n^{-c\nu_3} n^{1/4} = n^{-c(1/4c+\varepsilon)+1/4} = n^{-c\varepsilon} = o(1).$$

Part *xvii*) : By A6: $c < 1/4(D+1)$ let $c = 1/4(D+1) - \varepsilon$,

$$\begin{aligned}\log(n) n^{-1} h_1^{-D} h_2^{-(D+2)} n^{1/2} &\sim \log(n) n^{-1/2} n^{2(D+1)c} = \log(n) n^{-1/2+2(D+1)(1/4(D+1)-\varepsilon)} \\ &= n^{-1/2+1/2-2(D+1)\varepsilon} = n^{-2(D+1)\varepsilon} = o(1).\end{aligned}$$

Part *xviii*) : By (*xiv*) and (*xv*), $M_{2n} = \left[\frac{\log(n)}{nh_1^D h_2^D}\right]^{1/2} + h_1^{\nu_3} + h_2^{\nu_3} = o(n^{-1/4}) + o(n^{-1/4})$.

Part *xix*) : By A6: $a > 1/2k$, and $c < (2k-1)/[4k(D+1)]$, let $a = 1/2k + \varepsilon$ and $c = (2k-1)/[4k(D+1)] - \varepsilon$,

$$\begin{aligned}l_n n^{-1} h_1^{-D} h_2^{-(D+2)} &\sim n^a n^{-1} n^{2c(D+1)} \\ &= n^{1/2k+\varepsilon-1+2(D+1)(2k-1)/[4k(D+1)]-2(D+1)\varepsilon} \\ &= n^{1/2k-1+1-1/2k-(2D+1)\varepsilon} \\ &= n^{-(2D+1)\varepsilon} = o(1).\end{aligned}$$

Parts *xx*) - *xxv*) : Proofs follow, mutatis mutandis, from parts *xix*) - *xxiv*) are mutatis mutandis virtually identical to the proofs of parts *xii*) - *xiv*) please consult them.

Parts *xxvi*) & *xxvii*) By the above results,

$$M_n = M_{1n} + M_{2n} = o(n^{-1/4}), \qquad \mathcal{L}_{0n} = L_n + M_n = o(n^{-1/4}), \qquad \mathcal{L}_{1n} = L_n + M_n + N_{1n} = o(n^{-1/4}),$$

$$\mathcal{L}_{1n} = L_n + M_n + N_{2n} + b_2^{-1}\left(\sqrt{\frac{l_n}{n}} + l_n^{-k}\right) = o(n^{-1/4}), \qquad \mathcal{L}_n = \mathcal{L}_{1n} + \mathcal{L}_{2n} = o(n^{-1/4}).$$

□

The following is Lemma B.1 of Ozabaci (2015), I include it here for completeness.

Lemma 4. *Under Assumption A1*

i.) $||Q_{n,BB} - Q_{BB}||^2 = O_p(l_n^2/n)$.

ii.) $\lambda_{\min}(Q_{nBB}) = \lambda_{\min}(Q_{BB}) + o_p(1)$.

iii.) $\lambda_{\max}(Q_{nBB}) = \lambda_{\max}(Q_{BB}) + o_p(1)$.

iv.) $||Q_{nBB}^{-1} - Q_{BB}^{-1}||_{sp} = O_p(l_n/n^{1/2})$.

Proof. Here I state Weyl's Inequality (Bernstein (2005) Thm. 8.4.11) and all relevant conclusions are drawn from it. Let A, B be matrices of order $(m \times m)$, where $\{\lambda_i(A)\}_{i=1}^m$ and $\{\lambda_i(B)\}_{i=1}^m$ be partial orderings of the eigenvalues of A and B respectively where, $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_m(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_m(B)$. Then for any $h \in \{1, 2, \ldots, m\}$,

$$\lambda_h(A) + \lambda_m(B) \leq \lambda_h(A+B) \leq \lambda_h(A) + \lambda_1(B).$$

Note also that for a symmetric matrix A, $|\lambda_{max}(A)|^2 = \lambda_{max}(AA') \leq ||A||^2$ and that $\{-\lambda_i(A)\}_{i=1}^m$ are the eigenvalues of $-A$. Accordingly, $\lambda_{\max}(-A) = \lambda_{\min}(A)$, $\lambda_{\min}(-A) = \lambda_{\max}(A)$ with this the following conclusions are drawn,

$$\text{if} \quad \lambda_{\min}(A) \geq 0 \quad \text{then} \quad \lambda_{\min}(A) \geq -\lambda_{\max}(A) = -|\lambda_{max}(A)| \geq -||A||,$$

$$\text{if} \quad \lambda_{\min}(A) \le 0 \quad \text{then} \quad 0 \le -\lambda_{\min}(A) = \lambda_{max}(-A) \le ||-A|| = ||A||.$$

Consequently $0 \ge \lambda_{\min}(A) \ge -\lambda_{\max}(-A) \ge -||A||$.

Part *i.*) : Let $\sum_{a,c,L,J} \equiv \sum_{a=1}^{q}\sum_{L=1}^{l_n+2k}\sum_{c=1}^{q}\sum_{J=1}^{l_n+2k}$, by Assumptions A1.ii and A3.i,

$$\begin{aligned}
E\Big(||Q_{nBB} - Q_{BB}||^2\Big) &= E\left(||n^{-1}\sum_{i=1}^{n}\mathbf{B}_n(W_i)\mathbf{B}_n(W_i)' - E[\mathbf{B}_n(W_i)\mathbf{B}_n(W_i)']||^2\right) \\
&= \sum_{a=1}^{q}\sum_{L=1}^{l_n+2k}\sum_{c=1}^{q}\sum_{J=1}^{l_n+2k} E\left(\left[n^{-1}\sum_{i=1}^{n}B_{Lk}(W_{ai})B_{Jk}(W_{ci}) - E\left[B_{Lk}(W_{ai})B_{Jk}(W_{ci})\right]\right]^2\right) \\
&= \sum_{a,c,L,J} n^{-2}\Bigg\{\sum_{i=1}^{n}E\Big(B_{Lk}(W_{ai})B_{Jk}(W_{ci}) - E\Big[B_{Lk}(W_{ai})B_{Jk}(W_{ci})\Big]\Big)^2 \\
&\quad + \sum_{i=1}^{n}\sum_{l\neq i}^{n}E\Big(B_{Lk}(W_{ai})B_{Jk}(W_{ci})B_{Lk}(W_{al})B_{Jk}(W_{cl}) - B_{Lk}(W_{ai})B_{Jk}(W_{ci})E\Big[B_{Lk}(W_{al})B_{Jk}(W_{cl})\Big]\Big) \\
&\quad - \sum_{i=1}^{n}\sum_{l\neq i}^{n}E\left(E\Big[B_{Lk}(W_{ai})B_{Jk}(W_{ci})\Big]B_{Lk}(W_{al})B_{Jk}(W_{cl}) - E\Big[B_{Lk}(W_{ai})B_{Jk}(W_{ci})\Big]E\Big[B_{Lk}(W_{al})B_{Jk}(W_{cl})\Big]\right)\Bigg\} \\
&= \sum_{a,c,L,J} n^{-2}\sum_{i=1}^{n}E\Big(B_{Lk}(W_{ai})B_{Jk}(W_{ci}) - E\Big[B_{Lk}(W_{ai})B_{Jk}(W_{ci})\Big]\Big)^2 \\
&= \sum_{a,c,L,J} n^{-1}V\Big(B_{Lk}(W_{ai})B_{Jk}(W_{ci})\Big) \\
&\le n^{-1}E\left(\sum_{a=1}^{q}\sum_{L=1}^{l_n+2k}B_{Lk}(W_{ai})^2\sum_{c=1}^{q}\sum_{J=1}^{l_n+2k}B_{Jk}(W_{ci})^2\right) \\
&\le n^{-1}q(l_n+2k)\Big(\max_{\substack{1\le c\le q \\ 1\le J\le l_n+2k}}\sup_{W_c\in G_{W_c}} B_{Jk}(W_c)^2\Big)E\left(\sum_{a=1}^{q}\sum_{L=1}^{l_n+2k}B_{Lk}(W_{ai})^2\right) \\
&= O\left(\frac{l_n}{n}\right)E\Big(\text{trace}\Big[\mathbf{B}_n(W_i)\mathbf{B}_n(W_i)'\Big]\Big) = O\left(\frac{l_n}{n}\right)\text{trace}(Q_{BB}) \\
&\le O\left(\frac{l_n^2}{n}\right)\lambda_{\max}(Q_{BB}) \\
&= O\left(\frac{l_n^2}{n}\right) = o(1).
\end{aligned}$$

The last inequality is due to Assumption A1. Then (i) follows from Markov's inequality.

Part *ii.*) Let $h = l_n$ so that $\lambda_h(\cdot) = \lambda_{\min}(\cdot)$,

$$\begin{aligned}
\lambda_{\min}(Q_{nBB}) &= \lambda_{\min}(Q_{BB} + Q_{nBB} - Q_{BB}) \\
&\le \lambda_{\min}(Q_{BB}) + \lambda_{\max}(Q_{nBB} - Q_{BB}) \\
&\le \lambda_{\min}(Q_{BB}) + ||Q_{nBB} - Q_{BB}|| \\
&= \lambda_{\min}(Q_{BB}) + O_p(l_n n^{-1/2}).
\end{aligned}$$

By Assumption A1, $O_p(l_n^2 n^{-1}) = o_p(1)$, so that $\lambda_{\min}(Q_{nBB}) \le \lambda_{\min}(Q_{BB}) + o_p(1)$.

$$\begin{aligned}
\lambda_{\min}(Q_{BB}) &= \lambda_{\min}(Q_{BB} + Q_{nBB} - Q_{BB}) \\
&\ge \lambda_{\min}(Q_{BB}) + \lambda_{\min}(Q_{nBB} - Q_{BB}) \\
&\ge \lambda_{\min}(Q_{BB}) - ||Q_{nBB} - Q_{BB}|| \\
&= \lambda_{\min}(Q_{BB}) - O_p(l_n n^{-1/2})
\end{aligned}$$

$$= \lambda_{\min}(Q_{BB}) - o_p(1).$$

Combining the previous two statement gives (ii),

$$\lambda_{\min}(Q_{BB}) + o_p(1) \geq \lambda_{\min}(Q_{nBB}) \geq \lambda_{\min}(Q_{BB}) - o_p(1).$$

Part *iii.*) Let $h = 1$ so that $\lambda_h(\cdot) = \lambda_{\max}(\cdot)$.

$$\begin{aligned}\lambda_{\max}(Q_{nBB}) &= \lambda_{\max}(Q_{BB} + Q_{nBB} - Q_{BB}) \\ &\geq \lambda_{\max}(Q_{BB}) + \lambda_{\min}(Q_{nBB} - Q_{BB}) \\ &\geq \lambda_{\max}(Q_{BB}) - ||Q_{nBB} - Q_{BB}|| \\ &= \lambda_{\max}(Q_{BB}) - o_p(1).\end{aligned}$$

Furthermore,

$$\begin{aligned}\lambda_{\max}(Q_{nBB}) &= \lambda_{\max}(Q_{BB} + Q_{nBB} - Q_{BB}) \\ &\leq \lambda_{\max}(Q_{BB}) + \lambda_{\max}(Q_{nBB} - Q_{BB}) \\ &\leq \lambda_{\max}(Q_{BB}) + ||Q_{nBB} - Q_{BB}|| \\ &= \lambda_{\max}(Q_{BB}) + o_p(1).\end{aligned}$$

Combining the previous two statements gives (iii)

$$\lambda_{\max}(Q_{BB}) + o_p(1) \geq \lambda_{\max}(Q_{nBB}) \geq \lambda_{\max}(Q_{BB}) - o_p(1).$$

Part *iv.*) By the sub-multiplicative property of the spectral norm, and symmetry of Q_{nBB} and Q_{BB},

$$\begin{aligned}||Q_{nBB}^{-1} - Q_{BB}^{-1}||_{sp} &= ||Q_{nBB}^{-1}\left(Q_{BB} - Q_{nBB}\right)Q_{BB}^{-1}||_{sp} \\ &\leq ||Q_{nBB}^{-1}||_{sp}||Q_{BB} - Q_{nBB}||_{sp}||Q_{BB}^{-1}||_{sp} \\ &= \lambda_{\max}(Q_{nBB}^{-1}Q_{nBB}^{-1})^{1/2}\lambda_{\max}\left[(Q_{BB} - Q_{nBB})(Q_{BB} - Q_{nBB})\right]^{1/2}\lambda_{\max}(Q_{BB}^{-1}Q_{BB}^{-1})^{1/2} \\ &\leq \lambda_{\max}(Q_{nBB}^{-1})\lambda_{\max}(Q_{BB} - Q_{nBB})\lambda_{\max}(Q_{BB}^{-1}) \\ &\leq \lambda_{\min}(Q_{nBB})^{-1}||Q_{BB} - Q_{nBB}||\lambda_{\min}(Q_{BB})^{-1} \\ &= (c_{bl} + o_p(1))^{-1}O_p(l_n n^{-1/2})c_{bl}^{-1} = O_p(l_n n^{-1/2}).\end{aligned}$$

□

Lemma 5. *By Assumptions A1, A2, A3, and A4 for any sequence of random variables* $\{A_i\}_{i=1}^n$ *where* $\sup_{W,V_d \in G_{W,V_d}} E(A_i^2|W_i, V_{di}) \leq C \leq \infty$.

i.) $||\,(nh_1^D h_2^{D+1})^{-1}\mathbf{D}_d K_{3i}'\boldsymbol{B}_n\,||_E = O_p(1)$.

ii.) $||\,n^{-1}\mathbf{B}_n'\mathbf{V}_d\,||_E = O_p\left(l_n^{1/2}n^{-1/2}\right)$.

iii.) $||\,n^{-1}\mathbf{B}_n'(\mathbf{M}_d - \mathbf{M}_d^{l_n})\,||_E = O_p\left(l_n^{-k}\right)$.

iv.) $||\,[(n-1)b_2^2]^{-1}\mathbf{K}_2^{(1)}(V_d)'\dot{\Theta}_2^d\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n\,||_E = O_p\left(b_2^{-1}\right)$.

v.) $||\,[(n-1)b_2^2]^{-1}\phi_i u_i\mathbf{K}_2^{(1)}(V_d)'\dot{\Theta}_2^d\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n\,||_E = O_p\left(b_2^{-1}\right)$.

vi.) $||\,[(n-1)b_2^2]^{-1}\phi_i\zeta_{ci}\mathbf{K}_2^{(1)}(V_d)'\dot{\Theta}_2^d\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n\,||_E = O_p\left(b_2^{-1}\right)$.

where $\dot{\mathbf{A}} = diag\left(\{A_i\}_{i=1}^n\right)$.

Proof. First, by A2 and A3

$$E\left[(h_1^D h_2^D)^{-1}D_d K_{3ji}^2\middle|W_j, S_i\right] = E\left[(h_1^D h_2^D)^{-1}D_d K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}]^2\middle|W_j, S_i\right]$$

$$= (h_1^D h_2^D)^{-1} \int D_d K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] p(X_j, V_j, W_j) p(W_j)^{-1} dX_j dV_j$$

$$= (h_1^D h_2^D)^{-1} \int D_d K_3[(\gamma_1; \gamma_2)]^2 p(X_i + \gamma_1 h_1, V_i + h_2\gamma_2, W_j) p(W_j)^{-1} h_1^D d\gamma_1 h_2^D d\gamma_2$$

$$\leq \sup_{X,V,W \in G_{XVW}} p(X, V, W) \Big(\inf_{W \in G_W} p(W) \Big)^{-1} \int D_d K_3[(\gamma_1', \gamma_2')']^2 d\gamma_1 d\gamma_2 = O(1).$$

Consequently, uniformly in $G_{X,V}$,

$$E\Big[(h_1^D h_2^D)^{-1} D_d K_{3ji}^2 \Big| W_j, S_i\Big] = O(1).$$

By Assumption A2, $\lim_{\gamma_1, \gamma_2 \to \pm\infty} K_3[(\gamma_1', \gamma_2')] \equiv K[(\pm\infty, \pm\infty)] = 0$ then by A2 and A4

$$\Big|E\Big[(h_1^D h_2^{D+1})^{-1} D_d K_{3ji} \Big| W_j, S_i\Big]\Big| = \Big|E\Big[(h_1^D h_2^{D+1})^{-1} D_d K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] \Big| W_j, S_i\Big]\Big|$$

$$= \Big|(h_1^D h_2^{D+1})^{-1} \int D_d K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] p(X_j, V_j, W_j) p(W_j)^{-1} dX_j dV_j\Big|$$

$$\leq \Big|(h_1^D h_2^{D+1})^{-1} \Big[K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}] p(X_j, V_j, W_j) p(W_j)^{-1}\Big]_{-\infty}^{\infty}\Big|$$

$$+ \Big|(h_1^D h_2^{D+1})^{-1} \int h_2 K_3[H^{-1}\{(X_j', V_j')' - (X_i', V_i')'\}\}] D_d p(X_j, V_j, W_j) p(W_j)^{-1} dX_j dV_j\Big|$$

$$\leq \sup_{X,V,W \in G_{XVW}} p(X, V, W) \Big(\inf_{W \in G_W} p(W) \Big)^{-1} (h_1^D h_2^{D+1})^{-1} \Big|K_3[(\infty, \infty)] - K_3[(-\infty, -\infty)]\Big|$$

$$+ \Big|(h_1^D h_2^D)^{-1} \int K_3[(\gamma_1', \gamma_2')'] D_d p(X_i + h_1\gamma_1, V_i + h_2\gamma_2, W_j) p(W_j)^{-1} h_1^D d\gamma_1 h_2^D d\gamma_2\Big|$$

$$\leq \sup_{X,V,W \in G_{XVW}} \big|D_d p(X, V, W)\big| \Big(\inf_{W \in G_W} p(W) \Big)^{-1} \int |K_3[(\gamma_1', \gamma_2')']| d\gamma_1 d\gamma_2 = O(1)$$

Consequently, uniformly in $G_{X,V}$,

$$\Big|E\Big[(h_1^D h_2^{D+1})^{-1} D_d K_{3ji} \Big| W_j, S_i\Big]\Big| = O(1).$$

Note that,

$$E\left(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\right) = \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} E\left(B_{J,k}(W_{ai})^2\right) = \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} E\left(\frac{b_{J,k}(W_a)^2}{||b_{J,k}(W_a)||_2^2}\right) = \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} 1 = q(l_n + 2k).$$

Let $|\mathbf{B}_n(W)|$ be an element wise absolute value. For $j \neq i$ note that,

$$E\big(|\mathbf{B}_n(W_i)'||\mathbf{B}_n(W_j)|\big) = \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} E\left(|B_{J,k}(W_{ai})|\right) E\left(|B_{J,k}(W_{aj})|\right) = \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} E\left(|B_{J,k}(W_{ai})|\right)^2$$

$$= \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} (||b_{J,k}(W_{ai})||_2^2)^{-1} \left(\int_{t_J}^{t_{J+k}} b_{J,k}(W_{ai}) p(W_{ai}) dW_{ai}\right)^2$$

$$= \max_{\substack{1 \leq a \leq k \\ 1 \leq J \leq l_n + 2k}} (||b_{J,k}(W_{ai})||_2)^{-2} \sup_{W_a \in G_{W_a}} |b_{j,k}(W_a)|^2 |p(W_a)|^2 \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} \left(\int_{t_J}^{t_{J+k}} dW_{ai}\right)^2$$

$$= O(l_n) \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} [t_J - t_{J+k}]^2 = O(l_n) \sum_{a=1}^{q} \sum_{J=1}^{l_n+2k} O(l_n^{-2}) = O(1).$$

Consequently for all $i \neq j$, $E\big(|\mathbf{B}_n(W_i)'||\mathbf{B}_n(W_j)|\big) = O(1)$, where $|\mathbf{B}_n(W)|$ be an element wise absolute value.

Also,

$$\begin{aligned}E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2\Big|W_j,S_i\Big] &= \int b_2^{-1}K_2^{(1)}[b_2^{-1}(V_{dj}-V_{di})]^2p(V_j,W_j)p(W_j)^{-1}dV_j\\ &= \int b_2^{-1}K_2^{(1)}(\gamma)^2p(V_{di}+b_2\gamma,W_j)p(W_j)^{-1}b_2d\gamma\\ &\leq \sup_{V_dW\in G_{V_d,W}} p(V_d,W)\Big(\inf_{W\in G_W}p(W)\Big)^{-1}\int K_2^{(1)}(\gamma)^2d\gamma\\ &= O(1).\end{aligned}$$

Consequently both of the following are true, $E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2\Big|W_j,S_i\Big]=O(1)$ and $E\Big[b_2^{-1}|K_{2ji}^{(1)}(V_d)|\Big|W_j,S_i\Big]=O(1)$.

Part *i.*),

$$\begin{aligned}&E\Big(||(nh_1^Dh_2^{D+1})^{-1}\mathbf{D}_dK_{3i}'\mathbf{B}_n||_E^2\Big)=E\Big(||(nh_1^Dh_2^{D+1})^{-1}\sum_{j=1}^n\mathbf{B}_n(W_i)D_dK_{3ji}||_E^2\Big)\\ &=E\Big((nh_1^Dh_2^{D+1})^{-2}\sum_{i=1}^n\sum_{g=1}^n\mathbf{B}_n(W_i)'\mathbf{B}_n(W_g)D_dK_{3ji}D_dK_{3gi}\Big)\\ &\leq (nh_1^Dh_2^{D+1})^{-2}\sum_{j=1}^nE\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)D_dK_{3ji}^2\Big)\\ &\qquad+\Big|(nh_1^Dh_2^{D+1})^{-2}\sum_{j=1}^n\sum_{g\neq j}^nE\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_g)D_dK_{3ji}D_dK_{2gi}\Big)\Big|\\ &=(n^2h_1^Dh_2^{D+2})^{-1}\sum_{j=1}^nE\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[(h_1^Dh_2^D)^{-1}D_dK_{3ji}^2\Big|W_j,S_i\Big]\Big)\\ &\qquad+n^{-2}\sum_{j=1}^n\sum_{g\neq j}^nE\Big(\big|\mathbf{B}_n(W_j)'\big|\big|\mathbf{B}_n(W_g)\big|\Big|E\Big[(h_1^Dh_2^{D+1})^{-1}D_dK_{3ji}\Big|W_j,S_i,S_g\Big]E\Big[(h_1^Dh_2^{D+1})^{-1}D_dK_{3gi}\Big|W_g,S_i,S_j\Big]\Big|\Big)\\ &\leq O([nh_1^Dh_2^{D+2}]^{-1})n^{-1}\sum_{j=1}^nE\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)\Big)+O(n^{-2})\sum_{j=1}^n\sum_{g=1}^nE\Big(|\mathbf{B}_n(W_j)'||\mathbf{B}_n(W_g)|\Big)\\ &=O\left(\frac{l_n}{nh_1^Dh_2^{D+2}}\right)+O(1)=o(1)+O(1)=O(1).\end{aligned}$$

The last inequality is due to Lemma 3, *i.*) follows from Markov's Inequality.

Part *ii.*),

$$\begin{aligned}&E\Big(||n^{-1}\mathbf{B}_n'\mathbf{V}_d||_E^2\Big)=E\Big(||n^{-1}\sum_{j=1}^n\mathbf{B}_n(W_j)V_{dj}||_E^2\Big)\\ &=E\Big(n^{-2}\sum_{j=1}^n\sum_{g=1}^n\mathbf{B}_n(W_j)'V_{dj}\mathbf{B}_n(W_g)V_{dg}\Big)\\ &=E\Big(n^{-2}\sum_{j=1}^n\mathbf{B}_n(W_i)'\mathbf{B}_n(W_j)V_{dj}^2\Big)+E\Big(n^{-2}\sum_{j=1}^n\sum_{g\neq j}^n\mathbf{B}_n(W_j)V_{dj}\mathbf{B}_n(W_g)V_{dg}\Big)\\ &=n^{-2}\sum_{j=1}^nE\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[V_{dj}^2\Big|W_j\Big]\Big)+n^{-2}\sum_{j=1}^n\sum_{g\neq j}^nE\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_g)E\Big[V_{dj}\Big|W_j,S_g\Big]E\Big[V_{dg}\Big|W_g,S_j\Big]\Big)\\ &=O(n^{-1})E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)\Big)=O(l_nn^{-1}).\end{aligned}$$

ii.) follows from Markov's Inequality.

Part *iii.*) Let $\psi \in \mathbb{R}^{q(l_n+2k)}$ and consider;

$$\begin{aligned}
||n^{-1}\mathbf{B}_n'(\mathbf{M}_{dn}-\mathbf{M}_{dn}^{l_n})||_E^2 &= E||n^{-1}\mathbf{B}_n'(\mathbf{M}_{dn}-\mathbf{M}_{dn}^{l_n})||_{sp}^2 \\
&= ||\ n^{-1}\sum_{j=1}^{n}\mathbf{B}_n(W_j)[m_d(W_j)-m_d^{l_n}(W_j)]\ ||_{sp}^2 \\
&= \lambda_{\max}\Big(n^{-1}\sum_{j=1}^{n}\sum_{g=1}^{n}\mathbf{B}_n(W_j)\mathbf{B}_n(W_g)'[m_d(W_j)-m_d^{l_n}(W_j)][m_d(W_g)-m_d^{l_n}(W_g)]\Big) \\
&= \max_{||\psi||_E=1} n^{-2}\sum_{j=1}^{n}\sum_{g=1}^{n}\psi'\mathbf{B}_n(W_j)\mathbf{B}_n(W_g)'\psi[m_d(W_j)-m_d^{l_n}(W_j)][m_d(W_g)-m_d^{l_n}(W_g)] \\
&\le \sup_{W\in G_w}|m_d(W)-m_d^{l_n}(W)|^2 \max_{||\psi||_E=1}\psi'(Q_{nBB})\psi \\
&\le O(l_n^{-2k})\lambda_{\max}(Q_{nBB})||\psi||_E^2 = O(l_n^{-2k})\Big(\lambda_{\max}(Q_{BB})+o_p(1)\Big) = O(l_n^{-2k})
\end{aligned}$$

iii.) follows by taking square roots.

Part *iv.*):

$$\begin{aligned}
&E\Big(||\ [(n-1)b_2^2]^{-1}\mathbf{K}_2^{(1)}(V_d)'\dot{\Theta}_{2n}^d\dot{\mathbf{A}}_nI_n(-i)\mathbf{B}_n\ ||_E^2\Big) = E\Big(|||[(n-1)b_2^2]^{-1}\sum_{j\ne i}^{n}\mathbf{B}_n(W_j)K_{2ji}^{(1)}(V_d)\theta_{2j}^dA_j||_E^2\Big) \\
&= E\Big([(n-1)b_2^2]^{-1}\sum_{j\ne i}^{n}\sum_{g\ne i}^{n}\mathbf{B}_n(W_i)'\mathbf{B}_n(W_g)\theta_{2j}^d\theta_{2g}^dK_{2ji}^{(1)}(V_d)K_{2gi}^{(1)}(V_d)A_jA_g\Big) \\
&\le \sup_{X,V\in G_{X,V}}\theta_2^d(X,V)^2\Big\{[(n-1)^2b_2^3]^{-1}\sum_{j\ne i}^{n}E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2E\Big(A_j^2\Big|W_j,V_{dj},S_i\Big)|W_j\Big]\Big) \\
&\quad + [(n-1)^2b_2]^{-2}\sum_{j\ne i}^{n}\sum_{\substack{g\ne i\\ g\ne j}}^{n}E\Big(|\mathbf{B}_n(W_j)'||\mathbf{B}_n(W_g)|E\Big[b_2^{-1}|K_{2ji}^{(1)}(V_d)|E\Big(|A_j|\Big|W_j,V_{dj},S_i,S_g\Big)\Big|W_j,S_i,S_g\Big]^2\Big)\Big\} \\
&\le O([(n-1)b_2^3]^{-1})E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2\Big|W_j\Big]\Big) \\
&\quad + O([(n-1)^2b_2]^{-2})\sum_{j\ne i}^{n}\sum_{\substack{g\ne i\\ g\ne j}}^{n}E\Big(|\mathbf{B}_n(W_j)'||\mathbf{B}_n(W_g)|E\Big[b_2^{-1}|K_{2ji}^{(1)}(V_d)|\Big|W_j,S_i,S_g\Big]^2\Big) \\
&= O([(n-1)b_2^3]^{-1})E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)\Big) + O(b_2^{-2})E\Big(|\mathbf{B}_n(W_j)'||\mathbf{B}_n(W_g)|\Big) \\
&= O(l_n[(n-1)b_2^3]^{-1}) + O(b_2^{-2}) \\
&= o(1) + O(b_2^{-2}).
\end{aligned}$$

Where the last equality is due to Lemma 3, *iv.*) follows by Markov's inequality.

Part *v.*),

$$\begin{aligned}
&E\Big(||\ [(n-1)b_2^2]^{-1}\phi_iu_i\mathbf{K}_2^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2j}^d\dot{\mathbf{A}}_nI_n(-i)\mathbf{B}_n||_E^2\Big) = E\Big(||\ [(n-1)b_2^2]^{-1}\phi_iu_i\sum_{j\ne i}^{n}\mathbf{B}_n(W_i)K_{2ji}^{(1)}(V_d)\theta_2^dA_j\ ||_E^2\Big) \\
&= E\Big(\phi_i^2E[u_i^2|X_i,V_i,S_{-i}][(n-1)^2b_2^2]^{-1}\sum_{j\ne i}^{n}\sum_{g\ne i}^{n}\mathbf{B}_n(W_j)'\mathbf{B}_n(W_g)\theta_{2j}^d\theta_{2g}^dK_{2ji}^{(1)}(V_d)K_{2gi}^{(1)}(V_d)A_jA_g\Big) \\
&= \sigma_u^2\sup_{X,V\in G_{XV}}\phi(X,V)^2\theta_2^d(X,V)^2\Big\{[(n-1)^2b_2^3]^{-1}\sum_{j\ne i}^{n}E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)b_2^{-1}K_{2ji}^{(1)}(V_d)^2A_j^2\Big)
\end{aligned}$$

$$+ [(n-1)^2 b_2^2]^{-1} \sum_{j\neq i}^{n} \sum_{\substack{g\neq i\\ g\neq j}}^{n} E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|\,|b_2^{-1}K_{2ji}^{(1)}(V_d)|\,|b_2^{-1}K_{2gi}^{(1)}(V_d)|\,|A_j|\,|A_g|\Big)\Big\}$$

$$= O([(n-1)b_2^3]^{-1})E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2 E\Big(A_j^2\Big|W_j, V_{dj}, S_i\Big)\Big|W_j, S_i\Big]\Big)$$

$$+ O(b_2^{-2})E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|E\Big[|b_2^{-1}K_{2ji}^{(1)}(V_d)|E\Big(|A_j|\Big|W_j, V_{dj}, S_{-j}\Big)\Big|W_j, S_{-j}\Big]^2\Big)$$

$$= O([(n-1)b_2^3]^{-1})E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2\Big|W_j, S_i\Big]\Big)$$

$$+ O(b_2^{-2})E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|E\Big[|b_2^{-1}K_{2ji}^{(1)}(V_d)|\Big|W_j, S_{-j}\Big]^2\Big)$$

$$= O([(n-1)b_2^3]^{-1})E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)\Big) + O(b_2^{-2})E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|\Big)$$

$$= O\left(\frac{l_n}{(n-1)b_2^3}\right) + O\left(b_2^{-2}\right) = o(1) + O\left(b_2^{-2}\right)$$

Where the last equality if due to Lemma 3, *v.*) follows by Markov's inequality.

Part *vi.*),

$$E\Big(||[(n-1)b_2^2]^{-1}\phi_i\zeta_{ci}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\mathbf{\Theta}}_2^d\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E^2\Big) = E\Big(||[(n-1)b_2^2]^{-1}\phi_i\zeta_{ci}\sum_{j\neq i}^{n}\mathbf{B}_n(W_i)K_{2ji}^{(1)}(V_d)\theta_{2j}^d A_j||_E^2\Big)$$

$$= E\Big([(n-1)^2 b_2^3]^{-1}\phi_i^2\zeta_{ci}^2\sum_{j\neq i}^{n}\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)b_2^{-1}K_{2ji}^{(1)}(V_d)^2\left[\theta_{2j}^d\right]^2 A_j^2\Big)$$

$$+ E\Big([(n-1)^2 b_2^2]^{-1}\phi_i^2\zeta_{ci}^2\sum_{j\neq i}^{n}\sum_{\substack{g\neq i\\ g\neq j}}^{n}\mathbf{B}_n(W_j)'b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d\mathbf{B}_n(W_g)b_2^{-1}K_{2gi}^{(1)}(V_d)\theta_{2g}^d\Big)$$

$$\leq \sup_{X,V\in G_{XV}}\theta_2(X,V)^2\Big\{E\Big(E\big[\phi_i^2\zeta_{ci}^2\big|V_{di}, S_{-i}\big][(n-1)^2 b_2^3]^{-1}\sum_{j\neq i}^{n}\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)b_2^{-1}K_{2ji}^{(1)}(V_d)^2 A_j^2\Big)$$

$$+ E\Big(E\big[\phi_i^2\zeta_{ci}^2|V_{di}, S_{-i}\big][(n-1)^2 b_2^2]^{-1}\sum_{j\neq i}^{n}\sum_{\substack{g\neq i\\ g\neq j}}^{n}\Big|\mathbf{B}_n(W_j)'b_2^{-1}K_{2ji}^{(1)}(V_d)\mathbf{B}_n(W_g)b_2^{-1}K_{2gi}^{(1)}(V_d)\Big|\Big)\Big\}$$

$$\leq O([(n-1)^2 b_2^3]^{-1})\sum_{j\neq i}^{n}E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2 E\Big(A_j^2\Big|W_j, V_{dj}, S_i\Big)\Big|W_j\Big]\Big)$$

$$+ O([(n-1)^2 b_2^2]^{-1})\sum_{j\neq i}^{n}\sum_{\substack{g\neq i\\ g\neq j}}^{n}E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|E\Big[|b_2^{-1}K_{2ji}^{(1)}(V_d)|E\Big(|A_j|\Big|W_j, V_{dj}, S_{-j}\Big)\Big|W_j, S_{-j}\Big]^2\Big)$$

$$\leq O([(n-1)^2 b_2^3]^{-1})\sum_{j\neq i}^{n}E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2\Big|W_j\Big]\Big)$$

$$+ O([(n-1)^2 b_2^2]^{-1})\sum_{j\neq i}^{n}\sum_{\substack{g\neq i\\ g\neq j}}^{n}E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|E\Big[|b_2^{-1}K_{2ji}^{(1)}(V_d)|\Big|W_j, S_{-j}\Big]^2\Big)$$

$$\leq O([(n-1)^2 b_2^3]^{-1})\sum_{j\neq i}^{n}E\Big(\mathbf{B}_n(W_j)'\mathbf{B}_n(W_j)\Big) + O([(n-1)^2 b_2^2]^{-1})\sum_{j\neq i}^{n}\sum_{\substack{g\neq i\\ g\neq j}}^{n}E\Big(|\mathbf{B}_n(W_j)'|\,|\mathbf{B}_n(W_g)|\Big)$$

$$= O\left(\frac{l_n}{(n-1)b_2^3}\right) + O(b_2^{-2}) = o(1) + O(b_2^{-2}).$$

Where the last equality is due to Lemma 3, *vi.*) follows by Markov's inequality.

□

Lemma 6. *Let $\{X_i, V_i, A_i\}_{i=1}^n$ be an i.i.d sequence of random variables, where A is real valued such that for some $a \geq 2$,*

$$\sup_{X_{di} \in G_{X_d}} E[|A_i|^a \mid X_{di}] \leq C < \infty \quad \text{and} \quad \sup_{V_{di} \in G_{V_d}} E[|A_i|^a \mid V_{di}] \leq C < \infty.$$

Then under the assumptions of this paper,

$$\sup_{V_{di} \in G_{V_d}} \left| [(n-1)b_2^2]^{-1} \sum_{j \neq i} \left[K_{2ji}^{(1)}(V_d)\theta_{2j}^d A_j - E\Big(K_{2ji}^{(1)}(V_d)\theta_{2j}^d A_j\Big)\right]\right| = O_p\left(\left[\frac{\log(n)}{(n-1)b_2^3}\right]^{1/2}\right)$$

$$\sup_{V_{di} \in G_{V_d}} \left| [(n-1)b_2]^{-1} \sum_{j \neq i} \left[K_{2ji}(V_d)\theta_{2j}^d A_j - E\Big(K_{2ji}(V_d)\theta_{2j}^d A_j\Big)\right]\right| = O_p\left(\left[\frac{\log(n)}{(n-1)b_2}\right]^{1/2}\right)$$

$$\sup_{X_{di} \in G_{X_d}} \left| [(n-1)b_1]^{-1} \sum_{j \neq i} \left[K_{1ji}(X_d)\theta_{1j}^d A_j - E\Big(K_{1ji}(X_d)\theta_{1j}^d A_j\Big)\right]\right| = O_p\left(\left[\frac{\log(n)}{(n-1)b_1}\right]^{1/2}\right).$$

Proof. Let $m \in \{0,1\}$ and define $C_s^{(m)}(V_{dj}; V_d) = b_2 K_2^{(m)}[b_2^{-1}(V_{dj} - V_d)]\theta_{2j}^d$ where $V_d \in G_{V_d}$ so that,

$$\begin{aligned} S_n(V_d) &= [(n-1)b_2^{m+1}]^{-1} \sum_{j \neq i} K_2^{(m)}[b_2^{-1}(V_{dj} - V_d)]\theta_{2j}^d A_j \\ &= [(n-1)b_2^{m+2}]^{-1} \sum_{j \neq i} b_2 K_2^{(m)}[b_2^{-1}(V_{dj} - V_d)]\theta_{2j}^d A_j \\ &= [(n-1)b_2^{m+2}]^{-1} \sum_{j \neq i} C_s^{(m)}(V_{dj}; V_d) A_j. \end{aligned}$$

By the Mean Value Theorem, for $V_d' \in G_{V_d}$ and some $\lambda \in (0,1)$,

$$\begin{aligned} \Big|C_s^{(m)}(V_{dj}; V_d) - C_s^{(m)}(V_{dj}; V_d')\Big| &= \Big|b_2\theta_{2j}^d\Big(K_2^{(m)}[b_2^{-1}(V_{dj} - V_d)] - K_2^{(m)}[b_2^{-1}(V_{dj} - V_d')]\Big)\Big| \\ &\leq b_2\theta_{2j}^d \Big|K_2^{(m+1)}\left[b_2^{-1}\lambda(V_{dj} - V_d) + b_2^{-1}(1-\lambda)(V_{dj} - V_d')\right]\Big| \Big|b_2^{-1}(V_d - V_d')\Big| \\ &\leq \sup_{\gamma \in \mathbb{R}} \big|K_2^{(m+1)}(\gamma)\big| \sup_{X,V \in G_{X,V}} \theta_2^d(X,V)\big|V_d - V_d'\big| \\ &\leq C\big|V_d - V_d'\big|. \end{aligned}$$

Let $\{B_n\}_{n=1}^\infty$ be a nondecreasing sequence of real positive numbers s.t. $\sum_{n=1}^\infty B_n^{-a} < \infty$ and define,

$$S_n^\tau(V_d) = [(n-1)b_2^{m+1}]^{-1} \sum_{j \neq i} C_s^{(m)}(V_{dj}; V_d) A_j 1\{|A_j| \leq B_n\}.$$

where $1\{\cdot\}$ is an indicator function. Consider,

$$\begin{aligned} \sup_{V_d \in G_{V_d}} \big|S_n(V_d) - E(S_n(V_d))\big| &\leq \sup_{V_d \in G_{V_d}} \big|S_n(V_d) - S_n^\tau(V_d)\big| + \sup_{V_d \in G_{V_d}} \big|E\big(S_n^\tau(V_d)\big) - E\big(S_n(V_d)\big)\big| \\ &\quad + \sup_{V_d \in G_{V_d}} \big|S^\tau(V_d) - E\big(S^\tau(V_d)\big)\big| \\ &\equiv T_1 + T_2 + T_3. \end{aligned}$$

T_1:

$$\begin{aligned} T_1 = \sup_{V_d \in G_{V_d}} \big|S_n(V_d) - S_n^\tau(V_d)\big| &= \sup_{V_d \in G_{V_d}} \left|[(n-1)b_2^{m+1}]^{-1} \sum_{j \neq i} C_s^{(m)}(V_{dj}; V_d) A_j \big(1 - 1\{|A_j| \leq B_n\}\big)\right| \\ &= \sup_{V_d \in G_{V_d}} \left|[(n-1)b_2^{m+1}]^{-1} \sum_{j \neq i} C_s^{(m)}(V_{dj}; V_d) A_j 1\{|A_j| > B_n\}\right|. \end{aligned}$$

Note that $\sum_{n=1}^{\infty} P(|A_j| > B_n) \leq \sum_{n=1}^{\infty} B_n^{-a} E(|A_j|^a) \leq C \sum_{n=1}^{\infty} B_n^{-a} < C < \infty$ so that by the Borel Cantelli lemma $P(\{|A_j| > B_n$ infinitely often$\}) = 0$ consequently there exist a $N_B \in \mathbb{N}$ s.t. $1\{|A_j| > B_n\} = 0$ whenever $n > N_B$. Consequently,

$$T_1 \leq \sup_{V_d \in G_{V_d}} \left| [(n-1)b_2^{m+1}]^{-1} \sum_{j \neq i} C_s^{(m)}(V_{dj}; V_d) A_j 1\{|A_j| > B_n\} \right| = 0 \quad \text{almost surely}$$

T_2 :

$$\begin{aligned} T_2 &= \Big| E\Big(S_n(V_d) - S_n^{\tau}(V_d)\Big)\Big| \leq E\Big([(n-1)b_2^{m+2}]^{-1} \sum_{j\neq i} b_2 |K_{2j}^{(m)}(V_d)| \theta_{2j}^d |A_j| 1\{|A_j| > B_n\}\Big) \\ &\leq \sup_{X,V \in G_{X,V}} \theta_2^d(X,V)[(n-1)b_2^{m+1}]^{-1} \sum_{j\neq i} E\Big(|K_{2ji}^{(m)}(V_d)| E\Big[|A_j| 1\{|A_j| > B_n\}\big| V_{dj}, S_i\Big]\Big) \\ &\leq O(1)[(n-1)b_2^{m+1}]^{-1} \sum_{j\neq i} \int |K_{2j}^{(m)}(V_d)| \int |A_j| 1\{|A_j| > B_n\} p(A_j|V_{dj}) dA_j p(V_{dj}) dV_{dj}. \end{aligned}$$

By Markov's and Hölder's inequalities,

$$\begin{aligned} &\int |A_j| 1\{|A_j| > B_n\} p(A_j|V_{dj}) dA_j \leq \Big(\int |A_j|^a p(A_j|V_{dj}) dA_j\Big)^{1/a} \Big(\int 1\{|A_j| > B_n\} p(A_j|V_{dj}) dA_j\Big)^{1-1/a} \\ &\quad = E\big[|A_j|\big|V_{dj}\big]^{1/a} P\big(|A_j| > B_n\big)^{1-1/a} \leq C\Big(B_n^{-a} E\Big[E\Big(|A_j|^a|V_{dj}\Big)\Big]\Big)^{1-1/a} \leq C B_n^{1-a}. \end{aligned}$$

Consequently,

$$\begin{aligned} T_2 &\leq C B_n^{1-a}[(n-1)b_2^{m+1}]^{-1} \sum_{j\neq i} \int \big|K_2^{(m)}[b_2^{-1}(V_{dj} - V_d)]\big| p(V_{dj}) dV_{dj} \\ &\leq O(B_n^{1-a} b_2^{-m}) \int b_2^{-1} \big|K_2^{(m)}(\gamma)\big| p(V_d + \gamma b_2) b_2 d\gamma \leq O(B_n^{1-a} b_2^{-m}) \sup_{\gamma \in \mathbb{R}} p(\gamma) \int \big|K_2^{(m)}(\gamma)\big| d\gamma \\ &= O(B_n^{1-a} b_2^{-m}). \end{aligned}$$

For any $V \in \mathbb{R}$ define $B(V; r) = \{V_d \in \mathbb{R} : |V - V_d| \leq r\}$. By Assumption A3, there exists a $V_{do} \in G_{Vd}$ and a $r_o \in \mathbb{R}$ s.t. $G_{V_d} \subset B(V_{do}; r_o)$ so that for all $V_d, V_d' \in G_{V_d}$, $|V_d - V_d'| \leq r_o$. Let $\{\delta_n\}_{n=1}^n$ be a nonincreasing sequence of positive real numbers s.t. $\lim_{n\to\infty} \delta_n = 0$, By the Heine-Borel Theorem, for all n there exists a finite collection $N(\delta_n) = \{V_d^l\}_{l=1}^{\rho_n}$ where for all $V_d \in G_{V_d}$ there exists a $V_d^l \in N(\delta_n)$ s.t. $|V_d - V_d^l| \leq \delta_n$. Consequently there exists $C \in \mathbb{R}$ s.t. $\rho_n = C\delta_n^{-1}$.

T_3:

$$\begin{aligned} T_3 &= \sup_{V_d \in G_{V_d}} |S^{\tau}(V_d) - E\big(S^{\tau}(V_d)\big)| \\ &\leq \sup_{V_d \in G_{V_d}} |S^{\tau}(V_d) - S^{\tau}(V_d^l)| + \sup_{V_d \in G_{V_d}} |E\Big(S^{\tau}(V_d^l) - S^{\tau}(V_d)\Big)| + \max_{1\leq l\leq \rho_n} |S^{\tau}(V_d^l) - E\big(S^{\tau}(V_d^l)\big)| \\ &\equiv T_{31} + T_{32} + T_{33}. \end{aligned}$$

T_{31} :

$$\begin{aligned} \sup_{V_d \in G_{V_d}} |S^{\tau}(V_d) - S^{\tau}(V_d^l)| &\leq \sup_{V_d \in G_{V_d}} [(n-1)b_2^{m+2}]^{-1} \sum_{j\neq i} \big|C_s^{(m)}(V_{dj}; V_d) - C_s^{(m)}(V_{dj}; V_d^l)\big| \big|A_j\big| 1\{|A_j| \leq B_n\} \\ &\leq \sup_{V_d \in G_{V_d}} [(n-1)b_2^{m+2}]^{-1} \sum_{j\neq i} C\big|V_d - V_d^l\big| \big|A_j\big| 1\{|A_j| \leq B_n\} \\ &\leq O\left(\frac{\delta_n}{b_2^{m+2}}\right) (n-1)^{-1} \sum_{j\neq i} |A_j| \\ &= O\left(\frac{\delta_n}{b_2^{m+2}}\right) [E(|A_j|) + o_p(1)] = O_p\left(\frac{\delta_n}{b_2^{m+2}}\right). \end{aligned}$$

T_{32}:

$$\begin{aligned}\sup_{V_d\in G_{V_d}} |E\big(S^\tau(V_d^l)-S^\tau(V_d)\big)| &\le \sup_{V_d\in G_{V_d}}[(n-1)b_2^{m+2}]^{-1}\sum_{j\ne i}E\Big(\big|C_s^{(m)}(V_{dj};V_d^l)-C_s^{(m)}(V_{dj};V_d)\big||A_j|1\{|A_j|\le B_n\}\Big)\\ &\le \sup_{V_d\in G_{V_d}}[(n-1)b_2^{m+2}]^{-1}\sum_{j\ne i}E\Big(C\big|V_d^l-V_d\big||A_j|1\{|A_j|\le B_n\}\Big)\\ &\le O\left(\frac{\delta_n}{b_2^{m+2}}\right)(n-1)^{-1}\sum_{j\ne i}E(|A_j|)=O\left(\frac{\delta_n}{b_2^{m+2}}\right).\end{aligned}$$

T_{33}: Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of positive real numbers s.t. $\lim_{n\to\infty}\varepsilon_n=0$, and note that,

$$\begin{aligned}&P\Big(\max_{1\le l\le\rho_n}|S^\tau(V_d^l)-E\big(S^\tau(V_d^l)\big)|\ge\varepsilon_n\Big)\le P\Big(\bigcup_{l=1}^{\rho_n}\big\{|S^\tau(V_d^l)-E\big(S^\tau(V_d^l)\big)|\ge\varepsilon_n\big\}\Big)\\ &\le\sum_{l=1}^{\rho_n}P\Big(|S^\tau(V_d^l)-E\big(S^\tau(V_d^l)\big)|\ge\varepsilon_n\Big)\le\rho_n\max_{1\le l\le\rho_n}P\Big(|S^\tau(V_d^l)-E\big(S^\tau(V_d^l)\big)|\ge\varepsilon_n\Big).\end{aligned}$$

Furthermore,

$$\begin{aligned}S^\tau(V_d^l)-E\big(S^\tau(V_d^l)\big)&\equiv(n-1)^{-1}\sum_{j\ne i}Q_{jn}\\ &=(n-1)^{-1}\sum_{j\ne i}b_2^{-(m+2)}\Big(C_s^{(m)}(V_{dj};V_d^l)A_j1\{|A_j|\le B_n\}-E\Big[C_s^{(m)}(V_{dj};V_d^l)A_j1\{|A_j|\le B_n\}\Big]\Big)\\ &=(n-1)^{-1}\sum_{j\ne i}b_2^{-(m+2)}B_n\Big(|C_s^{(m)}(V_{dj};V_d^l)|+E\big(|C_s^{(m)}(V_{dj};V_d^l)|\big)\Big)\\ &=(n-1)^{-1}\sum_{j\ne i}b_2^{-(m+2)}B_nb_2\Big(|K_{2j}^{(m)}(V_d)|\theta_{2j}^d+E\big[|K_{2j}^{(m)}(V_d)|\theta_{2j}^d\big]\Big)\\ &=\sup_{\gamma\in\mathbb{R}}\big|K_2^{(m)}(\gamma)\big|\sup_{XV\in G_{XV}}\theta_2^d(X,V)(n-1)^{-1}\sum_{j\ne i}b_2^{-(m+1)}B_n\le(n-1)^{-1}\sum_{j\ne i}C\left(\frac{B_n}{b_2^{m+1}}\right).\end{aligned}$$

As a result, the following is a sufficient condition for the application of Bernstein's Inequality.

$$|Q_{jn}|\le C\left(\frac{B_n}{b_2^{m+1}}\right)\quad\text{implies}\quad E\big(|Q_{jn}|^p\big)\le C\left(\frac{B_n}{b_2^{m+1}}\right)^{p-2}E\big(|Q_{jn}|^2\big).$$

where $p\in\mathbb{N}$. By Bernstein's inequality,

$$\begin{aligned}P\Big(|S^\tau(V_d^l)-E\big(S^\tau(V_d^l)\big)|\ge\varepsilon_n\Big)&=P\Big(|\sum_{j\ne i}Q_{jn}|>(n-1)\varepsilon_n\Big)\\ &\le 2\exp\left[-\frac{1/2(n-1)^2\varepsilon_n^2}{\sum_{j\ne i}E\big(Q_{jn}^2\big)+1/3CB_nb_2^{-(m+1)}(n-1)\varepsilon_n}\right]\\ &\le 2\exp\left[-\frac{(n-1)\varepsilon_n^2}{2(n-1)^{-1}\sum_{j\ne i}E\big(Q_{jn}^2\big)+2/3CB_nb_2^{-(m+1)}\varepsilon_n}\right]\\ &\le 2\exp\left[-\frac{(n-1)\varepsilon_n^2}{2E\big(Q_{jn}^2\big)+CB_nb_2^{-(m+1)}\varepsilon_n}\right].\end{aligned}$$

For constants $\Delta_\varepsilon,\alpha,\beta\in\mathbb{R}^+$ let,

$$\varepsilon_n=\left(\frac{\log(n)}{(n-1)b_2^\alpha}\right)^{1/2}\Delta_\varepsilon,\quad\text{and}\quad\delta_n=\left(\frac{n}{b_2^\beta}\right)^{-1/2},\quad\text{so that}\quad\rho_n=C\left(\frac{n}{b_2^\beta}\right)^{1/2}.$$

Consequently,

$$
\begin{aligned}
P\Big(|S^{\tau}(V_d) - E\big(S^{\tau}(V_d^l)\big)| \geq \varepsilon_n\Big) &\leq 2\exp\left[-\frac{(n-1)\varepsilon_n^2}{2E\big(Q_{jn}^2\big) + CB_n b_2^{-(m+1)}\varepsilon_n}\right] \\
&= 2\exp\left[-\frac{\log(n)b_2^{-\alpha}\Delta_\varepsilon^2}{2E\big(Q_{jn}^2\big) + CB_n b_2^{-(m+1)}\log(n)^{1/2}[(n-1)b_2^{\alpha}]^{-1/2}\Delta_\varepsilon}\right] \\
&= 2\exp\left[-\frac{\log(n)\Delta_\varepsilon^2}{2b_2^{\alpha}E\big(Q_{jn}^2\big) + CB_n b_2^{\alpha-(m+1)}\log(n)^{1/2}[(n-1)b_2^{\alpha}]^{-1/2}\Delta_\varepsilon}\right] \\
&= 2\exp\left[-\log(n)^{1/2}\frac{\Delta_\varepsilon^2}{C(n)}\right] = 2n^{-\Delta_\varepsilon^2/C(n)}.
\end{aligned}
$$

where $C(n) = 2b_2^{\alpha}E\big(Q_{jn}^2\big) + CB_n b_2^{\alpha-(m+1)}\log(n)^{1/2}[(n-1)b_2^{\alpha}]^{-1/2}\Delta_\varepsilon$ Now,

$$
\begin{aligned}
E\big(Q_{jn}^2\big) &= V\big(b^{-(m+2)}C_s^{(m)}(V_{dj};V_d^l)A_j 1\{|A_j| \leq B_n\}\big) \\
&\leq E\Big(b^{-2(m+2)}C_s^{(m)}(V_{dj};V_d^l)^2 A_j^2 1\{|A_j| \leq B_n\}\Big) \\
&\leq E\Big(b^{-2(m+2)}C_s^{(m)}(V_{dj};V_d^l)^2 E\big[A_j^2|V_{dj}\big]\Big) \\
&\leq \sup_{V_d \in G_{V_d}} E\big[A_j^2|V_d\big]b^{-2(m+1)}\int\big|K_{2ji}^{(m)}(V_d)\big|\theta_{2j}^d p(X_i,V_i)dX_i dV_i \\
&= O\big(b_2^{-(2m+1)}\big)\int|K_2^{(m)}(\gamma)|d\gamma = O\big(b_2^{-(2m+1)}\big).
\end{aligned}
$$

Now choose, $\alpha = 2m+1$, and $B_n = \left(\frac{(n-1)b_2}{\log(n)}\right)^{1/2}$. Consequently,

$$
\begin{aligned}
C(n) &= 2b_2^{\alpha}E\big(Q_{jn}^2\big) + CB_n b_2^{\alpha-(m+1)}\left(\frac{\log(n)}{[(n-1)b_2^{\alpha}}\right)^{1/2}\Delta_\varepsilon \\
&= 2b_2^{2m+1}Cb_2^{-(2m+1)} + CB_n b_2^{2m+1-m-1}b_2^{-m}\left(\frac{\log(n)}{(n-1)b_2}\right)^{1/2}\Delta_\varepsilon \\
&= 2C + CB_n\left(\frac{\log(n)}{(n-1)b_2}\right)^{1/2}\Delta_\varepsilon = 2C + C\left(\frac{(n-1)b_2}{\log(n)}\right)^{1/2}\left(\frac{\log(n)}{(n-1)b_2}\right)^{1/2}\Delta_\varepsilon \\
&= 2C + C\Delta_\varepsilon.
\end{aligned}
$$

As a result there exists a $\Delta_\varepsilon \in \mathbb{R}^+$ such that $\Delta_\varepsilon^2/C(n) > 1$, which implies.

$$
\begin{aligned}
P\left(\max_{1\leq l\leq\rho_n}|S^{\tau}(V_d^l) - E\big(S^{\tau}(V_d^l)\big)| \geq \left(\frac{\log(n)}{(n-1)b_2^{2m+1}}\right)^{1/2}\Delta_\varepsilon\right) &\leq 2\rho_n n^{-\Delta_\varepsilon^2/C(n)} \\
&= 2Cn^{1/2}b_2^{-\beta/2}n^{-\Delta_\varepsilon^2/C(n)} \\
&\leq C\left[b_2^{\beta}n^{2\Delta_\varepsilon^2/C(n)-1}\right]^{-1/2} \\
&\leq C\left[b_2^{\beta}n^{(2-1)}\right]^{-1/2} \\
&= C\left[b_2^{\beta}n\right]^{-1/2}.
\end{aligned}
$$

Choose $\beta = 3$ then by Assumption A6,

$$
P\left(\max_{1\leq l\leq\rho_n}|S^{\tau}(V_d^l) - E\big(S^{\tau}(V_d^l)\big)| \geq \left(\frac{\log(n)}{(n-1)b_2^{2m+1}}\right)^{1/2}\Delta_\varepsilon\right) \leq O([b_2^3 n]^{-1/2}) = o(1).
$$

Consequently, by Markov's inequality,

$$T_{33} = \max_{i \le l \le \rho_n} |S^\tau(V_d^l) - E\big(S^\tau(V_d^l)\big)| = O_p\left(\left[\frac{\log(n)}{(n-1)b_2^{2m+1}}\right]^{1/2}\right).$$

in all, given that $b_2 = o(1)$,

$$T_3 = T_{31} + T_{32} + T_{33} = O_p\left(\left[\frac{\log(n)}{(n-1)b_2^{2m+1}}\right]^{1/2}\right).$$

Thus, since $a \ge 2$,

$$\sup_{V_d \in G_{V_d}} |S_n(V_d) - E\big(S_n(V_d)\big)| \le T_1 + T_2 + T_3 = O_p\left(\left[\frac{\log(n)}{(n-1)b_2^{2m+1}}\right]^{1/2}\right).$$

The proof of the second and third result of the lemma are trivial modifications of the preceding and as such are not provided. □

Lemma 7. *Let $f_1 \in \mathcal{F}_{\nu_1}$, $f_2 \in \mathcal{F}_{\nu_2}$, $\gamma \in \mathbb{R}$ and $\lambda \in (0,1)$, under the assumptions A1 - A6 of this paper,*

$$\begin{aligned}
i.)&\ E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\big(f_1(X_{dj}) - f_1(X_{di})\big)\big|S_i\big] = O\big(b_1^{\nu_1}\big).\\
ii.)&\ E\big[b_2^{-1}K_{2ji}(V_d)\theta_{2j}^d\big(f_2(V_{dj}) - f_2(V_{di})\big)\big|S_i\big] = O\big(b_2^{\nu_2}\big).\\
iii.)&\ E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\eta_{1ji}^d \mid S_i\big] = O(b_1^{\nu_1}).\\
iv.)&\ E\big[b_2^{-1}K_{2ji}(V_d)\theta_{2j}^d\eta_{2ji}^d \mid S_i\big] = O(b_1^{\nu_2}).
\end{aligned}$$

Proof. :

Part *i.*): By Assumption A4, one can write

$$f_1(X_{dj}) - f_1(X_{di}) = \sum_{m=1}^{\nu_1 - 1}(m!)^{-1}f_1^{(m)}(X_{di})\big[X_{dj} - X_{di}\big]^m + (\nu_1)^{-1}f_1^{(\nu_1)}(\tilde{X}_d)\big[X_{dj} - X_{di}\big]^{\nu_1},$$

where for some $\lambda \in (0,1)$, one has $\tilde{X}_d = \lambda X_{di} + (1-\lambda)X_{dj}$. Consequently,

$$\begin{aligned}
&E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d(f_1(X_{dj}) - f_1(X_{di}))|S_i\big]\\
&= b_1^{-1}\int K_1[b_1^{-1}(X_{dj} - X_{di})](f_1(X_{dj}) - f_1(X_{di}))\frac{g(X_{-dj}, V_j)}{p(X_j, V_j)}p(X_j, V_j)dX_j dV_j\\
&= b_1^{-1}\int K_1[b_1^{-1}(X_{dj} - X_{di})](f_1(X_{dj}) - f_1(X_{di}))dX_{dj}\\
&= b_1^{-1}\int K_1(\ \gamma\ )(f_1(X_{di} + \gamma b_1) - f_1(X_{di}))b_1 d\gamma\\
&= \sum_{m=1}^{\nu_1}(m!)^{-1}f_1^{(m)}(X_{di})\int K_1(\ \gamma\ )b_1^m\gamma^m d\gamma + (\nu_1!)^{-1}b_1^{\nu_1}\int K_1(\ \gamma\ )f_1^{(\nu_1)}(X_{di} + \lambda b_1\gamma)\gamma^{\nu_1}d\gamma\\
&= O(b_1^{\nu_1})\sup_{\gamma \in k\mathbb{R}}\big|f_1^{(\nu_1)}(\gamma)\big|\int\big|K_1(\ \gamma\ )\big||\gamma|^{\nu_1}d\gamma = O(b_1^{\nu_1}).
\end{aligned}$$

Part *ii.*): The proof of *ii.*) follows, mutatis mutandis, from the proof of *i.*).

Part *iii.*):

$$\begin{aligned}
E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\eta_{1ji}^d(c)|S_i\big] &= E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\big(E[Z_{cj}|X_j, V_j] - H(Z_{cj}; X_{dj})\big)|S_i\big]\\
&= E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d E[Z_{cj}|X_j, V_j]|S_i\big]\\
&\quad - E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\big(H(Z_{cj}; X_{dj}) - H(Z_{ci}; X_{di})\big)|S_i\big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d H(Z_{ci};X_{di})|S_i\big]\\
&\equiv T_1 - T_2 - T_3.
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d E[Z_{cj}|X_j,V_j]|S_i\big]\\
&= \int b_1^{-1}K_{1ji}(X_d)\frac{g(X_{-dj},V_j)}{p(X_j,V_j)}\int Z_{cj}\frac{p(Z_{cj},X_j,V_j)}{p(X_j,V_j)}dZ_{cj}p(X_j,V_i)dX_j,dV_j\\
&= \int b_1^{-1}K_{1ji}(X_d)\int \frac{g(X_j,V_j)}{p(X_j,V_j)}Z_{cj}\frac{p(Z_{cj},X_j,V_j)}{p(X_{dj})}dZ_{cj}dX_{-dj},dV_jdX_{dj}\\
&= \int b_1^{-1}K_{1ji}(X_d)\Big[H(Z_{cj};X_{dj}) - H(Z_{ci};X_{di}) + H(Z_{ci};X_{di})\Big]dX_{dj}\\
&= \int b_1^{-1}K_{1ji}(X_d)\Big[H(Z_{cj};X_{dj}) - H(Z_{ci};X_{di})\Big]dX_{dj} + H(Z_{ci};X_{di})\int b_1^{-1}K_{1ji}(X_d)dX_{dj}\\
&= O(b_1^{\nu_1}) + H(Z_{ci};X_{di}).
\end{aligned}
$$

The last equality is due to Assumption A4 and *i.*), T_2: By Assumption A4 and *i.*),

$$
T_2 = E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\big(H(Z_{cj};X_{dj}) - H(Z_{ci};X_{di})\big)|S_i\big] = O(b_1^{\nu_1}).
$$

T_3:

$$
\begin{aligned}
E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d H(Z_{ci};X_{di})|S_i\big] &= H(Z_{ci};X_{di})E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d|S_i\big]\\
&= H(Z_{ci};X_{di})b_1^{-1}\int K_{1ji}(X_d)\frac{g(X_{-dj},V_j)}{p(X_j,V_j)}p(X_j,V_j)dX_jdV_j\\
&= H(Z_{ci};X_{di})b_1^{-1}\int K_{1ji}(X_d)dX_{dj} = H(Z_{ci};X_{di}).
\end{aligned}
$$

In summary,

$$
E\big[b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\eta_{1ji}^d|S_i\big] = T_1 - T_2 - T_3 = O(b_1^{\nu_1}) + H(Z_{ci};X_{di}) - O(b_1^{\nu_1}) - H(Z_{ci};X_{di}) = O(b_1^{\nu_1}).
$$

Part *iv.*): The proof of *iv.*) follows, mutatis mutandis, from the proof of (*iii*).

□

Lemma 8. *Under the assumptions A1 - A6,*

$$
\begin{aligned}
&i)\ \big|E\Big(\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})B_L(W_{ai})\Big)\big| = O(l_n^{-3/2})\\
&ii)\ \big|E\Big(\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})V_{di}B_L(W_{ai})B_J(W_{li})\Big)\big| = O(l_n^{-1})\\
&iii)\ E\Big(\phi_1^2\zeta_{ci}^2H_2^{(1)d}(Z_{mi})^2B_L(W_{ai})^2\Big)^{1/2} = O(l_n^{-1/2})\\
&iv)\ E\Big(\phi_i^2\zeta_{ci}^2H_2^{(1)d}(Z_{mi})^2V_{di}^2B_L(W_{ai})^2B_J(W_{li})^2\Big)^{1/2} = O(1)
\end{aligned}
$$

where $c,m \in \{1,2,\cdots,p\}$ *and* $a,l \in \{1,2,\dots,q\}$.

Proof. Let $(\Omega,\mathcal{A},P)$ be relevant probability space to this paper and define the following random vectors,

$$
\begin{aligned}
\Gamma(\omega;X,V) &= [X(\omega)'\ V(\omega)']' : \Omega \to \mathbb{R}^{2D}\\
\Gamma(\omega;Z_c,X,V) &= [Z_c(\omega)\ X(\omega)'\ V(\omega)']' : \Omega \to \mathbb{R}^{2D+1}\\
\Gamma(\omega;X,V,W_a) &= [X(\omega)'\ V(\omega)'\ W_a(\omega)]' : \Omega \to \mathbb{R}^{2D+1}\\
\Gamma(\omega;Z_c,X,V,W_a) &= [Z_c(\omega)\ X(\omega)'\ V(\omega)'\ W_a(\omega)]' : \Omega \to \mathbb{R}^{2D+2}.
\end{aligned}
$$

Now define the following sets,

$$
A_a(L;l_n) = \{\omega \in \Omega : B_L(W_a(\omega)) > 0\}
$$

$$
\begin{aligned}
B_a(L;l_n) &= W_a(A_a(L;l_n)) \subset \mathbb{R} \\
C_a(L;l_n) &= \Gamma(A_a(L;l_n);X,V,W_a) \subset \mathbb{R}^{2D+1} \\
D_{ac}(L;l_n) &= \Gamma(A_a(L;l_n);Z_c,X,V,W_a) \subset \mathbb{R}^{2D+2} \\
E(L;l_n) &= \Gamma(A_a(L;l_n);X,V) \subset \mathbb{R}^{2D} \\
F_c(L;l_n) &= \Gamma(A_a(L;l_n);Z_c,X,V) \subset \mathbb{R}^{2D+1}.
\end{aligned}
$$

Note that $C_a(L;l_n) \subset E(L;l_n) \times B_a(L;l_n)$, and $D_{ac}(L;l_n) \subset F_c(L;l_n) \times B_a(L;l_n)$. As a result for any functions $f(Z_c,X,V,W_a) : \mathbb{R}^{2D+2} \to \mathbb{R}^+$, and $f(X,V,W_a) : \mathbb{R}^{2D+1} \to \mathbb{R}^+$,

$$
\begin{aligned}
&\int_{D_{ac}(L;l_n)} f(Z_c,X,V,W_a)p(Z_c,X,V,W_a)d(Z_c,X,V,W_a) \\
&\qquad \leq \int_{F_c(L;l_n)\times B_a(L;l_n)} f(Z_c,X,V,W_a)p(Z_c,X,V,W_a)d(Z_c,X,V,W_a).
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{C_a(L;l_n)} f(X,V,W_a)p(X,V,W_a)d(X,V,W_a) \\
&\qquad \leq \int_{E(L;l_n)\times B_a(L;l_n)} f(X,V,W_a)p(X,V,W_a)d(X,V,W_a).
\end{aligned}
$$

Furthermore by the remarks to assumption A1,

$$
\int_{B_a(L;l_n)} dW_a = \big[W_a\big]_{t_L}^{t_{L+k}} = (t_{L+k} - t_L) \leq Ckl_n^{-1}.
$$

Consider,

$$
\begin{aligned}
P\big(B_L(W_{ai}) > 0\big) &= P\big(W_a \in B_a(L;l_n)\big) = \int_{B_a(L;l_n)} p(W_a)dW_a = P\big(\{\omega \in \Omega : W_a(\omega) \in B_a(L;l_n)\}\big) \\
&= P(A_a(L;l_n)) = P\big(\{\omega \in \Omega : [Z_c(\omega)\ X(\omega)'\ V(\omega)'\ ]' \in F_c(L;l_n)\}\big) \\
&= P\big(F_c(L;l_n)\big) = \int_{F_c(L;l_n)} p(Z_c,X,V)d(Z_c,X,V).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P\big(B_L(W_{ai}) > 0\big) &= P\big(A_a(L;l_n)\big) = P\big(\{\omega \in \Omega : [X(\omega)'\ V(\omega)']' \in E(L;l_n)\}\big) \\
&= P\big(E(L;l_n)\big) = \int_{E(L;l_n)} p(X,V)dXdV.
\end{aligned}
$$

Consequently,

$$
P\big(B_L(W_a) > 0\big) = \int_{B_a(L;l_n)} p(W_a)dW_a = \int_{F_c(L;l_n)} p(Z_c,X,V)dZ_c dX dV = \int_{E(L;l_n)} p(X,V)dX, dV.
$$

Now consider,

$$
\int_{C_a(L;l_n)} p(W_a)dW_a \leq \sup_{W_a \in G_{W_a}} p(W_a) \int_{C_a(L;l_n)} dW_a = Ckl_n^{-1} = O(l_n^{-1}).
$$

Consequently,

$$
0 \leq \int_{F_c(L;l_n)} p(Z_c,X,V)dZ_c dX dV = \int_{E(L;l_n)} p(X,V)dX, dV = \int_{B_a(L;l_n)} p(W_a)dW_a = O(l_n^{-1}).
$$

Hereafter notation is simplified in the following way, $B_a \equiv B_a(L;l_n)$, $C_a \equiv C_a(L;l_n)$, $D_{ac} \equiv D_{ac}(L;l_n)$, $E \equiv E(L;l_n)$, and $F_c \equiv F_c(L;l_n)$.

Part *i.*):
Case 1: $Z_c \neq W_a$,

$$
\begin{aligned}
&\big|E\big(\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})B_L(W_{ai})\big)\big| \leq E\big(\big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|B_L(W_{ai})1\big\{B_L(W_{ai})>0\big\}\big) \\
&\quad= \int_{D_{ac}} \big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|B_L(W_{ai})p(Z_{ci},X_i,V_i,W_{ai})dZ_{ci}dX_idV_idW_{ai} \\
&\quad\leq \int_{F_c\times B_a} \big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|B_L(W_{ai})p(Z_{ci},X_i,V_i,W_{ai})dZ_{ci}dX_idV_idW_{ai} \\
&\quad= \int_{F_c} \big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|\int_{B_a} B_L(W_{ai})p(Z_{ci}X_i,V_i,W_{ai})p(Z_{ci},X_i,V_i)^{-1}dW_{ai}p(Z_{ci},X_i,V_i)dZ_{ci}dX_idV_i \\
&\quad\leq \big(||b_L(W_{ai})||_2\big)^{-1}\sup_{W_a\in G_{W_a}} b_L(W_a)\sup_{Z_c,X,V\in G_{Z_c,X,V,W_a}} p(Z_c,X,V,W_a)\big(\inf_{Z_c,X,V\in G_{Z_c,X,V}} p(Z_c,X,V)\big)^{-1} \\
&\qquad\qquad\times\int_{F_c} \big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|\int_{B_a} dW_{ai}\ p(Z_{ci},X_i,V_i)dZ_{ci}dX_idV_i \\
&\quad\leq O\big(l_n^{-1/2}\big)\sup_{X,V\in G_{XV}}\big|\phi(X,V)\zeta_cH_2^{(1)d}(Z_m)\big|\int_{F_c} p(Z_{ci},X_i,V_i)dZ_{ci}dX_idV_i \\
&\quad= O(l_n^{-3/2}).
\end{aligned}
$$

Case 2: $Z_c = W_a$,

$$
\begin{aligned}
&\big|E\big(\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})B_L(W_{ai})\big)\big| \leq E\Big(\phi_i\big|W_{ai}-H^*(W_{ai})\big|\big|H_2^{(1)d}(Z_{mi})\big|B_L(W_{ai})\Big) \\
&\quad\leq \int_{C_a} \phi_i\big|H_2^{(1)d}(Z_{mi})\big|\big|W_{ai}\big|B_L(W_{ai})p(X_i,V_i,W_{ai})dX_idV_idW_{ai} \\
&\qquad\quad+\int_{C_a} \phi_i\big|H_2^{(1)d}(Z_{mi})\big|\big|H^*(W_{ai})\big|B_L(W_{ai})p(X_i,V_i,W_{ai})dX_idV_idW_{ai} \\
&\quad\leq \int_{E} \phi_i\big|H_2^{(1)d}(Z_{mi})\big|\int_{B_a} \big|W_{ai}\big|B_L(W_{ai})p(X_i,V_i,W_{ai})p(X_i,V_i)^{-1}dW_{ai}\ p(X_i,V_i)dX_idV_i \\
&\qquad\quad+\int_{E} \phi_i\big|H_2^{(1)d}(Z_{mi})\big|\big|H^*(W_{ai})\big|\int_{B_a} B_L(W_{ai})p(X_i,V_i,W_{ai})p(X_i,V_i)^{-1}dW_{ai}\ p(X_i,V_i)dX_idV_i \\
&\quad\leq \big(||b_L(W_{ai})||_2\big)^{-1}\sup_{W_a\in G_{W_a}} b_L(W_a)\sup_{X,V\in G_{X,V,W_a}} p(X,V,W_a)|W_a|\big(\inf_{X,V\in G_{X,V}} p(X,V)\big)^{-1} \\
&\qquad\quad\times\left\{\int_{E} \phi_i\big|H_2^{(1)d}(Z_{mi})\big|\Big(1+\big|H^*(W_{ai})\big|\Big)\int_{B_a} dW_{ai}\ p(X_i,V_i)dX_idV_i\right\} \\
&\quad= O\big(l_n^{-1/2}\big)\sup_{X,V\in G_{XV}}\phi(X,V)\big|H_2^{(1)d}(Z_m)\big|\Big(1+\big|H^*(W_a)\big|\Big)\int_{E} p(X_i,V_i)dX_idV_i \\
&\quad= O(l_n^{-3/2}).
\end{aligned}
$$

Part *ii.*):

$$
\begin{aligned}
&\Big|E\Big(\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})V_{di}B_L(W_{ai})B_J(W_{li})\Big)\Big| \\
&\qquad\leq \big(||b_L(W_l)||_2\big)^{-1}\sup_{V_d,W_l\in G_{V_d,W_l}}\big|V_d\big|\big|b_L(W_l)\big|E\big(\big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|B_L(W_{ai})1\big\{B_L(W_{ai})>0\big\}\big) \\
&\qquad= O(l_n^{1/2})E\big(\big|\phi_i\zeta_{ci}H_2^{(1)d}(Z_{mi})\big|B_L(W_{ai})1\big\{B_L(W_{ai})>0\big\}\big) \\
&\qquad= O(l_n^{1/2})O(l_n^{-3/2}) = O(l_n^{-1}).
\end{aligned}
$$

Part *iii.*):

$$
E\Big(\phi_1^2\zeta_{ci}^2H_2^{(1)d}(Z_{mi})^2B_L(W_{ai})^2\Big)^{1/2} \leq \big(||b_L(W_{ai})||_2\big)^{-1/2}\sup_{W_a\in G_{W_a}} b_L(W_a)^{1/2}E\Big(\phi_1^2\zeta_{ci}^2H_2^{(1)d}(Z_{mi})^2B_L(W_{ai})\Big)^{1/2}
$$

$$\leq O(l_n^{1/4})E\Big(\phi_1^2\zeta_{ci}^2 H_2^{(1)d}(Z_{mi})^2 B_L(W_{ai})\Big)^{1/2}.$$

Note that a careful inspection of the proof of Part (i), yields the following conclusion,

$$E\Big(\phi_1^2\zeta_{ci}^2 H_2^{(1)d}(Z_{mi})^2 B_L(W_{ai})\Big) = O(l_n^{-3/2}).$$

Consequently, $E\Big(\phi_1^2\zeta_{ci}^2 H_2^{(1)d}(Z_{mi})^2 B_L(W_{ai})^2\Big)^{1/2} = O(l_n^{1/4})O(l_n^{-3/4}) = o(l_n^{-1/2})$.

Part *iv.*):

$$\begin{aligned}
&E\Big(\phi_i^2\zeta_{ci}^2 H_2^{(1)d}(Z_{mi})^2 V_{di}^2 B_L(W_{ai})^2 B_J(W_{li})^2\Big)^{1/2} \\
&\quad\leq \big(||b_L(W_{li})||_2\big)^{-1}\big(||b_L(W_{ai})||_2\big)^{-1/2} \sup_{W_a,W_l\in G_{W_a,W_l}} b_L(W_l)b_L(W_a)^{1/2}E\Big(\phi_1^2\zeta_{ci}^2 H_2^{(1)d}(Z_{mi})^2 V_{di}^2 B_L(W_{ai})\Big)^{1/2} \\
&\quad= O(l_n^{1/2})O(l_n^{1/4})\sup_{V_d\in G_{V_d}}|V_d| E\Big(\phi_1^2\zeta_{ci}^2 H_2^{(1)d}(Z_{mi})^2 B_L(W_{ai})\Big)^{1/2} = O(l_n^{3/4})O(l_n^{-3/4}).
\end{aligned}$$

□

Lemma 9. *Suppose that for some sequence of positive real numbers* $\{T_n\}_{n=1}^{\infty}$,

$$\sup_{X_{di}\in G_X}|\hat{p}(X_{di}) - p(X_{di})| = O_p(T_n), \quad \textit{and} \quad \sup_{V_{di}\in G_V}|\hat{p}(V_{di}) - p(V_{di})| = O_p(T_n).$$

Then, under the assumptions A1 - A6,

$$\hat{g}(X_i,\hat{V}_i) - g(X_i,V_i) = 2\sum_{d=1}^{D} g(X_{-di},V_i)\big[\hat{p}(X_{di}) - p(X_{di})\big] + 2\sum_{d=1}^{D} g(X_i,V_{-di})\big[\hat{p}(\hat{V}_{di}) - p(V_{di})\big] + O_p(T_n^2).$$

Proof. Let $T = [T_1\ T_2\ \cdots T_D]'$ be a $D\in\mathbb{N}$ dimensional random vector, $p(T_d)$ be the density of each of its components having compact support G_{T_d} and such that,

$$\sup_{T_d\in G_{T_d}}|\hat{p}(T_d) - p(T_d)| = O_p(L_n) \quad \text{and} \quad \sup_{T_d\in G_{T_d}} p(T_d) = C < \infty.$$

For the purposes of this proof, D can be any natural number, consequently this proof will be carried out algorithmically. Define,

$$A_j = \prod_{j\leq d}^{D}\hat{p}(T_d) - \prod_{j\leq d}^{D} p(T_d), \qquad \text{and} \qquad B_j = \hat{p}(T_j) - p(T_j) + p(T_j).$$

Define $S = \{s\in\mathbb{N} : 1\leq s\leq D\}$, then for any $\Theta_i\subset S$ s.t. $\#(\Theta_i) = \text{card}(\Theta_i) = i$ define $\Phi_i = \Theta_i^c\cap S$ and,

$$D(\Theta_i) = \Big[\prod_{d\in\Phi_i} p(T_d)\Big]\Big[\prod_{d\in\Theta_i}\big(\hat{p}(T_d) - p(T_d)\big)\Big].$$

Furthermore for any $1\leq j\leq D-1$ define, $C(j,\{j\}) = \big(\hat{p}(T_j) - p(T_j)\big)\prod_{j+1\leq d}^{D} p(T_d)$. T hen, note that,

$$\begin{aligned}
A_j &= \prod_{j\leq d}^{D}\hat{p}(T_d) - \prod_{j\leq d}^{D} p(T_d) \\
&= \prod_{j\leq d}^{D}\hat{p}(T_d) - \hat{p}(T_j)\prod_{j+1\leq d}^{D} p(T_d) + \hat{p}(T_j)\prod_{j+1\leq d}^{D} p(T_d) - \prod_{j\leq d}^{D} p(T_d)
\end{aligned}$$

$$
\begin{aligned}
&= \hat{p}(T_j)\Big[\prod_{j+1\le d}^{D}\hat{p}(T_d) - \prod_{j+1\le d}^{D} p(T_d)\Big] + \big(\hat{p}(T_j) - p(T_j)\big)\prod_{j+1\le d}^{D} p(T_d)\\
&= \big(\hat{p}(T_j) - p(T_j) + p(T_j)\big)A_{j+1} + C(j,\{j\})\\
&= B_j A_{j+1} + C(j,\{j\}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
A_1 &= B_1A_2 + C(1,\{1\})\\
&= B_1\big(B_2A_3 + C(2,\{2\}) + C(1,\{1\})\big)\\
&= B_1B_2A_3 + B_1C(2,\{2\}) + C(1,\{1\})\\
&= B_1B_2\big(B_3A_4 + C(3,\{3\})\big) + B_1C(2,\{2\}) + C(1,\{1\})\\
&= B_1B_2B_3A_4 + B_1B_2C(3,\{3\}) + B_1C(2,\{2\}) + C(1\{1\})\\
&\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&= A_D\prod_{i=1}^{D-1}B_i + \sum_{j=1}^{D-1}C(j+1,\{j+1\})\prod_{1\le i\le j}B_i + C(1,\{1\}).
\end{aligned}
$$

Now note that,

$$
\begin{aligned}
A_D\prod_{i=1}^{D-1}B_i &= \Big([\hat{p}(T_1) - p(T_1)] + p(T_1)\Big)\Big([\hat{p}(T_2) - p(T_2)] + p(T_2)\Big)\\
&\qquad \times\cdots\times\Big(\big[\hat{p}(T_{D-1}) - p(T_{D-1})\big] + p(T_{D-1})\Big)\Big(\hat{p}(T_D) - p(T_D)\Big)
\end{aligned}
$$

If one expands this product into a sum consisting solely of terms of the form $D(\Theta_{j+1})$ then, for each $j \in \{1,2,\ldots,D-1\}$, there are exactly $\binom{D-1}{j}$ of these terms within this sum. Furthermore, for all Θ_i one has

$$
\begin{aligned}
D(\Theta_i) &= \Big[\prod_{d\in\Phi_i}p(T_d)\Big]\Big[\prod_{d\in\Theta_i}\big(\hat{p}(T_d) - p(T_d)\big)\Big]\\
&\le \Big[\prod_{d\in\Phi_i}\sup_{T_d\in G_{T_d}}p(T_d)\Big]\Big[\prod_{d\in\Theta_i}\sup_{T_d\in G_{T_d}}\big|\hat{p}(T_d) - p(T_d)\big|\Big]\\
&= O(1)O_p\big(L_n^{\#(\Theta_i)}\big) = O_P\big(L_n^i\big).
\end{aligned}
$$

Consequently one can write

$$
A_D\prod_{i=1}^{D-1}B_i = \sum_{j=1}^{D}D(\{j\}) + \sum_{j=1}^{D-1}\binom{D-1}{j}O_p(L_n^{j+1}) = \sum_{j=1}^{D}D(\{j\}) + O_p(L_n^2).
$$

Similarly,

$$
\begin{aligned}
C(j+1,\{j+1\})\prod_{1\le i\le j}B_i &= \Big([\hat{p}(T_1) - p(T_1)] + p(T_1)\Big)\Big([\hat{p}(T_2) - p(T_2)] + p(T_2)\Big)\\
&\qquad \times\cdots\times\Big([\hat{p}(T_j) - p(T_j)] + p(T_j)\Big)\Big[\big(\hat{p}(T_{j+1}) - p(T_{j+1})\big)\prod_{j+2\le d}^{D}p(T_d)\Big].
\end{aligned}
$$

Furthermore, if one where to expand this product into a sum consisting soley of terms of the form $D(\Theta_{m+1})$ then for each $m \in \{1,2,\ldots,j\}$, there are exactly, $\binom{j}{m}$ of these in the sum. Consequently, one can write,

$$
C(j+1,\{j+1\})\prod_{1\le i\le j}B_i = D(\{j+1\}) + \sum_{m=1}^{j}\binom{j}{m}O_p(L_n^{m+1}) = D(\{j+1\}) + O_p(L_n^2).
$$

As a result, one has,

$$\sum_{j=1}^{D-1} C(j+1,\{j+1\}) \prod_{1\le i\le j} B_i = \sum_{j=1}^{D-1} D(\{j+1\}) + O_p(L_n^2).$$

Lastly note that $C(1,\{1\}) = D(\{1\})$ so that,

$$\begin{aligned}
A_1 &= A_D \prod_{i=1}^{D-1} B_i + \sum_{j=1}^{D-1} C(j+1,\{j+1\}) \prod_{1\le i\le j} B_i + C(1,\{1\}) \\
&= \sum_{j=1}^{D} D(\{j\}) + O_p(L_n^2) + \sum_{j=1}^{D-1} D(\{j+1\}) + O_p(L_n^2) + D(\{1\}) \\
&= 2\sum_{j=1}^{D} D(\{j\}) + O_p(L_n^2).
\end{aligned}$$

□

Lemma 10. *Under the assumptions A1 - A6,* $n^{-1}\sum_{i=1}^n \zeta_i u_i(\hat\phi_i - \phi_i) = o_p(n^{-1/2})$.

Proof.

$$\begin{aligned}
\hat\phi(X_i,\hat V_i) - \phi(X_i,V_i) &= [\hat p(X_i,\hat V_i)p(X_i,V_i)]^{-1}\Big(p(X_i,V_i)\big[\hat g(X_i,\hat V_i) - g(X_i,V_i)\big] + g(X_i,V_i)\big[p(X_i,V_i) - \hat p(X_i,\hat V_i)\big]\Big) \\
&= \hat p(X_i,\hat V_i)^{-1}\big[\hat g(X_i,\hat V_i) - g(X_i,V_i)\big] + [\hat p(X_i,\hat V_i)p(X_i,V_i)]^{-1} g(X_i,V_i)\big[p(X_i,V_i) - \hat p(X_i,\hat V_i)\big] \\
&= \hat p(X_i,\hat V_i)^{-1}\big[\big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\big) + \phi_i\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big] \\
&= \big[p(X_i,V_i)^{-1} + [\hat p(X_i,\hat V_i)p(X_i,V_i)]^{-1}\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big] \\
&\qquad \times \big[\big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\big) + \phi_i\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big] \\
&\le \Big[p(X_i,V_i)^{-1} + \big(p(X_i,V_i)^2 + p(X_i,V_i)\big[\hat p(X_i,\hat V_i) - p(X_i,V_i)\big]\big)^{-1}\big[\hat p(X_i,\hat V_i) - p(X_i,V_i)\big]\Big] \\
&\qquad \times \big[\big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\big) + \phi_i\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big] \\
&\equiv \big[p(X_i,V_i)^{-1} + A^*\big]\big[\big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\big) + \phi_i\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big],
\end{aligned}$$

where,

$$A^* = \Big[\big(p(X_i,V_i)^2 + p(X_i,V_i)\big[\hat p(X_i,\hat V_i) - p(X_i,V_i)\big]\big)^{-1}\big[\hat p(X_i,\hat V_i) - p(X_i,V_i)\big]\Big].$$

Furthermore,

$$\begin{aligned}
&A^*\big[\big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\big) + \phi_i\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big] \\
&\quad \le |A^*|\Big(\sup_{X,V\in G_{X,V}} |\hat g(X_i,\hat V_i) - g(X_i,V_i)| + \sup_{X,V\in G_{X,V}} p(X,V) \sup_{X,V\in G_{X,V}} |p(X_i,V_i) - \hat p(X_i,\hat V_i)|\Big) \\
&\quad = \big(p(X_i,V_i)^2 + p(X_i,V_i)\big[\hat p(X_i,\hat V_i) - p(X_i,V_i)\big]\big)^{-1}\big[\hat p(X_i,\hat V_i) - p(X_i,V_i)\big] O_p(\mathcal{L}_{0n}) \\
&\quad \le \Big(\inf_{X,V\in G_{X,V}} p(X_i,V_i)^2 + \inf_{X,V\in G_{X,V}} p(X_i,V_i) o_p(1)\Big)^{-1} \sup_{X,V\in G_{X,V}} \big|\hat p(X_i,\hat V_i) - p(X_i,V_i)\big| O_p(\mathcal{L}_{0n}) \\
&\quad = O_p(\mathcal{L}_{0n}^2) = o_p(n^{-1/2}).
\end{aligned}$$

Consequently,

$$\hat\phi(X_i,\hat V_i) - \phi(X_i,V_i) = p(X_i,V_i)^{-1}\big[\big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\big) + \phi_i\big(p(X_i,V_i) - \hat p(X_i,\hat V_i)\big)\big] + o_p(n^{-1/2})$$

By Lemmas 3 and 9,

$$p(X_i,V_i)^{-1}\Big(\hat g(X_i,\hat V_i) - g(X_i,V_i)\Big)$$

$$
\begin{aligned}
&= 2\sum_{d=1}^{D} p(X_i,V_i)^{-1}g(X_{-di},V_i)\big(\hat{p}(X_{di})-p(X_{di})\big)\\
&\qquad + 2\sum_{d=1}^{D} p(X_i,V_i)^{-1}g(X_i,V_{-di})\big(\hat{p}(\hat{V}_{di})-p(V_{di})\big) + O_p(\mathcal{L}_{0n}^2)\\
&= 2\sum_{d=1}^{D} p(X_{di})^{-1}\phi_i\big[\hat{p}(X_{di})-E(\hat{p}(X_{di})\big|X_{di})\big] + 2\sum_{d=1}^{D} p(X_{di})^{-1}\phi_i[E(\hat{p}(X_{di})|X_{di})-p(X_{di})]\\
&\qquad + 2\sum_{d=1}^{D} p(V_{di})^{-1}\phi_i\big[\hat{p}(\hat{V}_{di})-\hat{p}(V_{di})\big] + 2\sum_{d=1}^{D} p(V_{di})^{-1}\phi_i\big[\hat{p}(V_{di})-E(\hat{p}(V_{di})|V_{di})\big]\\
&\qquad + 2\sum_{d=1}^{D} p(V_{di})^{-1}\phi_i\big[E(\hat{p}(V_{di})|V_{di})-p(V_{di})\big] + o_p(n^{-1/2})\\
&\equiv 2\sum_{d=1}^{D}T_{1d} + 2\sum_{d=1}^{D}T_{2d} + 2\sum_{d=1}^{D}T_{3d} + 2\sum_{d=1}^{D}T_{4d} + 2\sum_{d=1}^{D}T_{5d} + o_p(n^{-1/2}).
\end{aligned}
$$

Consider also,

$$
\begin{aligned}
p(X_i,V_i)^{-1}\phi_i\big[p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big] &= (-1)p(X_i,V_i)^{-1}\phi_i\big[\hat{p}(X_i,\hat{V}_i)-\hat{p}(X_i,V_i)\big]\\
&\qquad - p(X_i,V_i)^{-1}\phi_i\big[\hat{p}(X_i,V_i)-E(\hat{p}(X_i,V_i)|X_i,V_i)\big]\\
&\qquad - p(X_i,V_i)^{-1}\phi_i\big[E(\hat{p}(X_i,V_i)|X_i,V_i\big]-p(X_i,V_i)\big]\\
&\equiv -T_6 - T_7 - T_8.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
n^{-1}\sum_{i=1}^{n}\zeta_i u_i(\hat{\phi}_i-\phi_i) &\le \sum_{d=1}^{D}2\Bigg\{n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_{1d} + n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_{2d} + n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_{3d} + n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_{4d}\\
&\qquad + n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_{5d} + n^{-1}\sum_{i=1}^{n}|\zeta_i||u_i|o_p(n^{-1/2})\Bigg\}\\
&\qquad - n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_6 - n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_7 - n^{-1}\sum_{i=1}^{n}\zeta_i u_i T_8\\
&= \sum_{d=1}^{D}\Big\{E_{1d}+E_{2d}+E_{3d}+E_{4d}+E_{5d}\Big\} - E_6 - E_7 - E_8 + o_p(n^{-1/2}).
\end{aligned}
$$

Note the last equality is due to Assumption A5 which implies $n^{-1}\sum_{i=1}^{n}|\zeta_i||u_i| = O_p(1)$,

$$
\begin{aligned}
E_{1d} &= n^{-1}\sum_{i=1}^{n}\zeta_i u_i p(X_{di})^{-1}\phi_i[nb_1]^{-1}\sum_{j=1}^{n}\big(K_{1ji}(X_d)-E\big[K_{1ji}(X_d)\big|X_{di}\big]\big)\\
&= n^{-2}\sum_{i=1}^{n}\zeta_i u_i p(X_{di})^{-1}\phi_i b_1^{-1}\big(K_{1ii}(X_d)-E\big[K_{1ii}(X_d)|X_{di}\big]\big)\\
&\qquad + n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\zeta_i u_i p(X_{di})^{-1}\phi_i b_1^{-1}\big(K_{1ji}(X_d)-E\big[K_{1ji}(X_d)|X_{di}\big]\big)\\
&= n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\zeta_i u_i p(X_{di})^{-1}\phi_i b_1^{-1}\big(K_{1ji}(X_d)-E\big[K_{1ji}(X_d)|X_{di}\big]\big),\\
&\equiv n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_1^d(i,j;c) \simeq \binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_1^{(2)d}(i,j;c) = U_1^{(2)d}(c).
\end{aligned}
$$

where, $\Gamma_1^{(2)d}(i,j;c) = \Psi_1^d(i,j;c) + \Psi_1^d(j,i;c)$

$$E\big(\Psi_1^d(i,j;c)\big) = E\Big(\zeta_{ci}p(X_{di})^{-1}\phi_i b_1^{-1}\big(K_{1ji}(X_d) - E\big[K_{1ji}(X_d)|X_{di}\big]\big)E\big[u_i|Z_i,X_i,V_i,S_j\big]\Big) = 0.$$

$$E\Big(E\big[\Psi_1^d(i,j;c)|S_i\big]^2\Big)^{1/2} = E\Big(p(X_{di})^{-2}\phi^2\zeta_{ci}^2u_i^2b_1^{-2}E\big[K_{1ji}(X_d) - E\big(K_{1ji}(X_d)|X_{di}\big)|S_i\big]^2\Big)^{1/2} = 0.$$

$$\begin{aligned}E\Big(E\big[\Psi_1^d(i,j;c)|S_j\big]^2\Big)^{1/2} &= E\Big(E\Big[p(X_{di})^{-1}\phi_i\zeta_{ci}b_1^{-1}\big(K_{1ji}(X_d) - E\big[K_{1ji}(X_d)|X_{di}\big]\big)E\big[u_i|Z_i,X_i,V_i,S_j\big]\Big|S_j\Big]^2\Big)^{1/2}\\ &= 0.\end{aligned}$$

$$\begin{aligned}E\big(\Psi_1^d(i,j;c)^2\big)^{1/2} &= E\Big(p(X_{di})^{-2}\phi_i^2\zeta_{ci}^2b_1^{-2}\big(K_{1ji}(X_d) - E\big[K_{1ji}(X_d)|X_{di}\big]\big)^2E\big[u_i^2|Z_i,X_i,V_i,S_j\big]\Big)^{1/2}\\ &= O(b_1^{-1/2})\sup_{X_d\in G_{X_d}} p(X_d)^{-1}E\Big(b_1^{-1}\big[K_{1ji}(X_d) - E\big(K_{1ji}(X_d)|X_{di}\big)\big]^2E\big[\phi_i^2\zeta_{ci}^2|X_{di},X_{dj}\big]\Big)\\ &= O(b_1^{-1/2})\Big(E\big(b_1^{-1}K_{1ji}(X_d)^2\big) - E\big[E\big(b_1^{-1/2}K_{1ji}(X_d)|X_{di}\big)^2\big]\Big)^{1/2}\\ &\le O(b_1^{-1/2})E\big(b_1^{-1}K_{1ji}(X_d)^2\big)^{1/2} = O\big(b_1^{-1/2}\big).\end{aligned}$$

Now in summary,

$$\begin{aligned}E_{1d} = U_1^{(2)d}(c) &= E\big(\Psi_1^d(i,j;c)\big) + O_p\Big(n^{-1/2}E\Big(E\big[\Psi_1^d(i,j;c)|S_i\big]^2\Big)^{1/2}\Big)\\ &\quad + O_p\Big(n^{-1/2}E\Big(E\big[\Psi_1^d(i,j;c)|S_j\big]^2\Big)^{1/2}\Big) + O_p\big(n^{-1}E\big(\Psi_1^d(i,j;c)^2\big)^{1/2}\big)\\ &= O_p\big([nb_1^{1/2}]^{-1}\big) = o_p(n^{1/2}).\end{aligned}$$

Recall that,

$$\begin{aligned}\sup_{X,V\in G_{X,V}}\big|E\big[\hat p(X_i,V_i)\big|X_i,V_i\big] - p(X_i,V_i)\big| &= O\big(h_1^{\nu_3} + h_2^{\nu_3}\big)\\ \sup_{X_{di}\in G_{X_d}}\big|E\big[\hat p(X_{di})\big|X_{di}\big] - p(X_{di})\big| &= O(h_0^{\nu_0})\\ \sup_{V_{di}\in G_{V_d}}\big|E\big[\hat p(V_{di})\big|V_{di}\big] - p(V_{di})\big| &= O(h_0^{\nu_0}).\end{aligned}$$

$$\begin{aligned}E\big(E_{2d}(c)^2\big) &= E\Big(\Big[n^{-1}\sum_{i=1}^n \zeta_{ci}u_ip(X_{di})^{-1}\phi_i\big(E\big[\hat p(X_{di})|X_{di}\big] - p(X_{di})\big)\Big]^2\Big)\\ &= n^{-2}\sum_{i=1}^n E\big(\zeta_{ci}^2u_i^2p(X_{di})^{-2}\phi_i^2\big(E\big[\hat p(X_{di})\big|X_{di}\big] - p(X_{di})\big)^2\big)\\ &\quad + n^{-2}\sum_{i=1}^n\sum_{j\neq i}E\Big[\phi_i\zeta_{ci}u_ip(X_{di})^{-1}\big(E\big[\hat p(X_{di})\big|X_{di}\big] - p(X_{di})\big)E\big(u_i|Z_i,X_i,V_i,S_{-i}\big)\\ &\qquad\qquad \times \phi_j\zeta_{cj}p(X_{dj})^{-1}\Big(E\big[\hat p(X_{dj})|X_{dj}\big] - p(X_{dj})\Big)E\big[u_j|Z_j,X_j,V_j,S_{-j}\big]\Big]\\ &\le n^{-1}\sup_{X_{di}\in G_{X_d}}\big|E\big[\hat p(X_{di})\big|X_{di}\big] - p(X_{di})\big|^2E\big(\zeta_{ci}^2\phi_i^2E\big[u_i^2|Z_i,X_i,V_i\big]\big)\\ &= O(n^{-1}b_1^{2\nu_0})E\big(\phi_i^2\zeta_{ci}^2\big) = O(n^{-1}b_1^{2\nu_0}).\end{aligned}$$

$$E_{3d}(c) = n^{-1}\sum_{i=1}^n p(V_{di})^{-1}\phi_i\zeta_{ci}u_i\big[\hat p(\hat V_{di}) - \hat p(V_{di})\big]$$

$$
\begin{aligned}
&= n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[nb_2]^{-1}\sum_{j=1}^{n}\Big(K_2[b_2^{-1}(\hat V_{dj}-\hat V_{di})]-K_2[b_2^{-1}(V_{dj}-V_{di})]\Big)\\
&= n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i\Bigg\{\sum_{m=1}^{3}(m!)^{-1}[nb_2^{1+m}]^{-1}\sum_{j=1}^{n}K_{2ji}^{(m)}(V_d)\big[(\hat V_{dj}-V_{dj})-(\hat V_{di}-V_{di})\big]^m\\
&\qquad\qquad + [4!nb_2^5]^{-1}\sum_{j=1}^{n}K_{2ji}^{(4)}(\tilde V_d)\big[(\hat V_{dj}-V_{dj})-(\hat V_{di}-V_{di})\big]^4\Bigg\}\\
&= (-1)n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[nb_2^2]^{-1}\sum_{j=1}^{n}K_{2ji}^{(1)}(V_d)(\hat m_d^{l_n}(W_j)-m_d(W_j))\\
&\quad + n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[nb_2^2]^{-1}\sum_{j=1}^{n}K_{2ji}^{(1)}(V_d)(\hat m_d^{l_n}(W_i)-m_d(W_i))\\
&\quad + n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[2nb_2^3]^{-1}\sum_{j=1}^{n}K_{2ji}^{(2)}(V_d)\big[(m_d(W_j)-\hat m_d^{l_n}(W_j))-(m_d(W_i)-\hat m_d^{l_n}(W_i))\big]^2\\
&\quad + n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[6nb_2^4]^{-1}\sum_{j=1}^{n}K_{2ji}^{(3)}(V_d)\big[(m_d(W_j)-\hat m_d^{l_n}(W_j))-(m_d(W_i)-\hat m_d^{l_n}(W_i))\big]^3\\
&\quad + n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[24nb_2^5]^{-1}\sum_{j=1}^{n}K_{2ji}^{(4)}(\tilde V_d)\big[(m_d(W_j)-\hat m_d^{l_n}(W_j))-(m_d(W_i)-\hat m_d^{l_n}(W_i))\big]^4\\
&\equiv -E_{31d}(c)+E_{32d}(c)+E_{33d}(c)+E_{34d}(c)+E_{35d}(c).
\end{aligned}
$$

Let $\alpha\in\{2,3\}$,

$$
\begin{aligned}
E\Big(|p(V_{di})^{-1}\phi_i||\zeta_{ci}||u_i||b_2^{-1}K_{2ji}^{(\alpha)}(V_d)|\Big) &= E\Big(p(V_{di})^{-1}\phi_i|u_i||b_2^{-1}K_{2ji}^{(\alpha)}(V_d)|\Big)\\
&\le \big[\inf_{V_d\in G_{V_d}} p(V_d)\big]^{-1}E\Big(|\phi_i\zeta_{ci}||u_i|E\big[|K_{2ji}^{(\alpha)}(V_d)|\big|S_i\big]\Big)\\
&= O(1)E\Big(|\phi_i\zeta_{ci}|E\big[|u_i|\big|Z_i,X_i,V_i\big]\Big) = O(1)E\Big(|\phi_i\zeta_{ci}|\Big)=O(1).
\end{aligned}
$$

Also,

$$
\begin{aligned}
E\Big(p(V_{di})^{-1}\phi_i|\zeta_{ci}||u_i||K_{2ji}^{(4)d}(\tilde V_d)|\Big) &\le \big[\inf_{V_d\in G_{V_d}} p(V_d)\big]^{-1}E\Big(|\phi_i\zeta_{ci}||u_i||b_2^{-1}K_{2ji}^{(4)}(V_d)|\Big)\\
&\le O(1)\sup_{\gamma\in\mathbb{R}}|K_2^{(4)}(\gamma)|E\Big(|\phi_i\zeta_{ci}|E\big[|u_i|\big|Z_i,V_i,X_i\big]\Big)\\
&\le O(1)E\Big(|\phi_i\zeta_{ci}|\Big)=O(1).
\end{aligned}
$$

Consequently by Markov's Inequality,

$$
|p(V_{di})^{-1}\phi_i\zeta_{ci}u_ib_2^{-1}K_{2ji}^{(2)}(V_d)|=O_p(1),\quad |p(V_{di})^{-1}\phi_i\zeta_{ci}u_ib_2^{-1}K_{2ji}^{(3)}(V_d)|=O_p(1),\quad |p(V_{di})^{-1}\phi_i\zeta_{ci}u_iK_{2ji}^{(4)}(\tilde V_d)|=O_p(1).
$$

$$
\begin{aligned}
&E_{33d}+E_{34d}+E_{35d}\\
&\quad= n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[2nb_2^3]^{-1}\sum_{j=1}^{n}K_{2ji}^{(2)}(V_d)\big[(m_d(W_j)-\hat m_d^{l_n}(W_j))-(m_d(W_i)-\hat m_d^{l_n}(W_i))\big]^2\\
&\qquad + n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[6nb_2^4]^{-1}\sum_{j=1}^{n}K_{2ji}^{(3)}(V_d)\big[(m_d(W_j)-\hat m_d^{l_n}(W_j))-(m_d(W_i)-\hat m_d^{l_n}(W_i))\big]^3\\
&\qquad + n^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\phi_i\zeta_{ci}u_i[24nb_2^5]^{-1}\sum_{j=1}^{n}K_{2ji}^{(4)}(\tilde V_d)\big[(m_d(W_j)-\hat m_d^{l_n}(W_j))-(m_d(W_i)-\hat m_d^{l_n}(W_i))\big]^4
\end{aligned}
$$

$$\le [2b_2^2]^{-1}\big[2\sup_{W\in G_W}|\hat{m}_d^{l_n}(W)-m_d(W)|\big]^2 n^{-2}\sum_{i=1}^n\sum_{j=1}^n |p(V_{di})^{-1}\phi_i\zeta_{ci}u_i b_2^{-1}K_{2ji}^{(2)}(V_d)|$$

$$+[6b_2^3]^{-1}\big[2\sup_{W\in G_W}|\hat{m}_d^{l_n}(W)-m_d(W)|\big]^3 n^{-2}\sum_{i=1}^n\sum_{j=1}^n |p(V_{di})^{-1}\phi_i\zeta_{ci}u_i b_2^{-1}K_{2ji}^{(3)}(V_d)|$$

$$+[24b_2^5]^{-1}\big[2\sup_{W\in G_W}|\hat{m}_d^{l_n}(W)-m_d(W)|\big]^4 n^{-2}\sum_{i=1}^n\sum_{j=1}^n |p(V_{di})^{-1}\phi_i\zeta_{ci}u_i K_{2ji}^{(4)}(\tilde{V}_d)|$$

$$= O_p\left(\frac{L_n^2}{b_2^2}\right)+O_p\left(\frac{L_n^3}{b_2^3}\right)+O_p\left(\frac{L_n^4}{b_2^5}\right)=o_p(n^{-1/2}).$$

By Lemma 3 vi.) and vii.) . Note by assumption A5 of this paper,

$$E\Big(\zeta_{ci}^2u_i^2\Big)=E\Big(\zeta_{ci}^2E\big[u_i^2|Z_i,X_i,V_i\big]\Big)=E(\zeta_{ci}^2)=O(1).$$

and

$$E\big(\zeta_{ci}^2u_i^2\big|W_i,V_{di}\big)=E\big(\zeta_{ci}^2E\big[u_i^2\big|W_i,X_i,V_i\big]|W_i,V_{di}\big)=O(1)E\big(\zeta_{ci}^2|W_i,V_{di}\big)=O(1).$$

Consequently, by a trivial modification of Lemma 6,

$$\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i\neq j}\Big(\theta_{2i}^d\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)-E\Big[\theta_{2i}^d\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)\Big]\Big)\Big|=O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right).$$

$$E_{31d}(c)=n^{-1}\sum_{i=1}^n\theta_{2i}^d\zeta_{ci}u_i[nb_2^2]^{-1}\sum_{j=1}^nK_{2ji}^{(1)}(V_d)\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)$$

$$=[n^2b_2^2]^{-1}\sum_{j=1}^n\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)p(V_{di})^{-1}\phi_i\zeta_{cj}u_jK_{2jj}^{(1)}(V_d)$$

$$+n^{-1}\sum_{j=1}^n\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)[nb_2^2]^{-1}\sum_{i\neq j}p(V_{di})^{-1}\phi_i\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)$$

$$\le\sup_{W\in G_W}|\hat{m}_d^{l_n}(W_i)-m_d(W_i)||K_2^{(1)}(b_2^{-1}0)|[nb_2]^{-2}\sum_{j=1}^n|\theta_{2j}^d\zeta_{ci}||u_i|$$

$$+\sup_{W\in G_W}|\hat{m}_d^{l_n}(W_i)-m_d(W_i)|\Big(n^{-1}\sum_{j=1}^n\Big|[nb_2^2]^{-1}\sum_{i\neq j}\Big(\theta_{2i}^d\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)-E\Big[\theta_{2i}^d\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)\Big]\Big)\Big|$$

$$+\Big|[nb_2^2]^{-1}\sum_{i\neq j}E\Big[\theta_{2i}^d\zeta_{ci}K_{2ji}^{(1)}(V_d)E\big[u_i\big|Z_i,X_i,V_i,S_j\big]\Big]\Big|\Big)$$

$$\le O\left(\frac{L_n}{nb_2^2}\right)+O_p(L_n)\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i\neq j}\Big(\theta_{2i}^d\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)-E\Big[\theta_{2i}^d\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)\Big]\Big)\Big|$$

$$=o_p(n^{-1/2})+O_p\left(L_n\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right)=o_p(n^{-1/2}).$$

By Lemma 3 vi) and xxiv)

Note that similar to the above,

$$E\Big(p(V_{di})^{-1}|\phi_i\zeta_{ci}||u_i||K_{2ii}^{(1)}(V_d)|\Big)\le\big[\inf_{V_d\in G_{V_d}}p(V_d)\big]^{-1}\sup_{\gamma\in\mathbb{R}}|K_2^{(1)}(\ \gamma\ )|E\Big(|\phi_i\zeta_{ci}|E\big[|u_i|\big|Z_i,V_i,X_i\big]\Big)=O(1).$$

So, by Markov's Inequality, $p(V_{di})^{-1}|\phi_i\zeta_{ci}||u_i||b_2^{-1}K_{2ii}^{(1)}(V_d)|=O_p(1)$. Now consider,

$$E_{32d}(c)=n^{-1}\sum_{i=1}^n\theta_{2i}^d\zeta_{ci}u_i[nb_2^2]^{-1}\sum_{j=1}^nK_{2ji}^{(1)}(V_d)\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)$$

$$
\begin{aligned}
&= [n^2b_2^2]^{-1}\sum_{i=1}^{n} p(V_{di})^{-1}\theta_{2i}^{d}\phi_i\zeta_{ci}u_iK_{2ii}^{(1)}(V_d)\big(\hat m_d^{l_n}(W_i)-m_d(W_i)\big)\\
&\qquad + n^{-1}\sum_{i=1}^{n}[nb_2^2]^{-1}\sum_{i\neq j} p(V_{di})^{-1}\phi_i\zeta_{ci}u_iK_{2ji}^{(1)}(V_d)\big(\hat m_d^{l_n}(W_j)-m_d(W_j)\big)\\
&\le \sup_{W\in G_w}|\hat m_d^{l_n}(W)-m_d(W)|[n^2b_2^2]^{-1}\sum_{i=1}^{n}p(V_{di})^{-1}|\phi_i\zeta_{ci}||u_i|K_{2ii}^{(1)}(V_d)\\
&\qquad + n^{-1}\sum_{j=1}^{n}[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)}(V_d)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\big[\hat{\mathbf{M}}_d^{l_n}-\mathbf{M}_d\big]\\
&\le O_p(L_n[nb_2^2]^{-1}) + n^{-1}\sum_{j=1}^{n}\big|\big|[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)d}(V_d)'\boldsymbol{\phi}_i\dot{\boldsymbol{\zeta}}_{cn}\dot{\boldsymbol{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n\big|\big|_E O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right)\\
&\qquad - n^{-1}\sum_{j=1}^{n}[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)}(V_d)'\boldsymbol{\phi}_i\dot{\boldsymbol{\zeta}}_{cn}\dot{\boldsymbol{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\big[\mathbf{M}_d^{l_n}-\mathbf{M}_d\big]\\
&= o_p(n^{-1/2}) + E_{321d}(c) + E_{322d}(c).
\end{aligned}
$$

$$
E_{321d}(c) = n^{-1}\sum_{j=1}^{n}||[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)}(V_d)'\boldsymbol{\phi}_n\dot{\mathbf{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n||_E O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right).
$$

Now, consider,

$$
\begin{aligned}
&E\Big(||[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)}(V_d)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n||_E^2\Big)\\
&\quad = E\Big(||[nb_2^2]^{-1}\sum_{i\neq j}\mathbf{B}_n(W_i)'K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}u_i||_E^2\Big)\\
&\quad = E\Big([n^2b_2^3]^{-1}\sum_{j\neq i}\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)b_2^{-1}K_{2ji}^{(1)}(V_d)^2\phi_i^2\zeta_{ci}^2p(V_{di})^{-2}u_i^2\Big)\\
&\qquad\quad + E\Big([n^2b_2^2]^{-1}\sum_{i\neq j}\sum_{\substack{g\neq i\\ g\neq j}}\mathbf{B}_n(W_i)'b_2^{-1}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}u_i\mathbf{B}_n(W_g)b_2^{-1}K_{2jg}^{(1)}(V_d)\phi_g\zeta_{cg}p(V_{dg})^{-1}u_g\Big)\\
&\quad \le \Big[\inf_{V_d\in G_{V_d}}p(V_d)\Big]^{-1}[n^2b_2^3]^{-1}\sum_{i\neq j}E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\phi_i^2\zeta_{ci}^2b_2^{-1}K_{2ji}^{(1)}(V_d)^2E\big[u_i^2|W_i,X_i,V_i\big]\Big)\\
&\qquad\quad + \sum_{i\neq j}^{n}E\Big(\mathbf{B}_n(W_i)'b_2^{-1}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}E\big[u_i|W_i,X_i,V_i,S_{-i}\big]\\
&\qquad\qquad\qquad \times\mathbf{B}_n(W_g)b_2^{-1}K_{2jg}^{(1)}(V_d)\phi_g\zeta_{cg}p(V_d)^{-1}E\big[u_g|W_g,X_g,V_g,S_{-g}\big]\Big)\\
&\quad = O([nb_2^3]^{-1})n^{-1}\sum_{i\neq j}E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\phi_i^2\zeta_{ci}^2E\big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2|S_i\big]\Big)\\
&\quad = O([nb_2^3]^{-1})n^{-1}\sum_{i\neq j}E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)E\big[\phi_i^2\zeta_{ci}^2|W_i\big]\Big)\\
&\quad = O([nb_2^3]^{-1})n^{-1}\sum_{i\neq j}E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\Big) = O(l_n/nb_2^3).
\end{aligned}
$$

Consequently, by Markov's Inequality,

$$
||[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)}(V_d)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n||_E = O_p\left(\left[\frac{l_n}{nb_2^3}\right]^{1/2}\right).
$$

Furthermore,

$$E_{321d}(c) = n^{-1}\sum_{j=1}^{n}||[nb_2^2]^{-1}\mathbf{K}_{2j}^{(1)}(V_d)'\boldsymbol{\phi}_n\dot{\mathbf{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n||_E O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right)$$
$$= O_p\left(\left[\frac{l_n}{nb_2^3}\right]^{1/2}\right)O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right) = O_p\left(\left[\frac{l_n}{nb_2^{3/2}}\right]\right) = o_p(n^{-1/2}).$$

Lemma 3 v).

$$E_{322d}(c) = n^{-1}\sum_{j=1}^{n}[nb_2^2]^{-1}\mathbf{K}_j^{(1)}(V_d)'\boldsymbol{\phi}_i\dot{\boldsymbol{\zeta}}_{cn}\dot{\boldsymbol{p}}(V_d)^{-1}\dot{\mathbf{u}}_nI_n(-j)\big[\mathbf{M}_d^{l_n}-\mathbf{M}_d\big]$$
$$= n^{-2}\sum_{j=1}^{n}\sum_{i\neq j}b_2^{-2}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}u_i\big[m_d^{l_n}(W_i)-m_d(W_i)\big]$$
$$= n^{-2}\sum_{j=1}^{n}\sum_{i\neq j}\Psi_{322}^d(j,i;c) \simeq \binom{n}{2}^{-1}\sum_{j=1}^{n}\sum_{j<i}\Gamma_{322}^{(2)d}(j,i;c) = U_{322}^{(2)d}(c)$$

where, $\Gamma_{322}^{(2)d}(j,i;c) = \Psi_{322}^d(j,i;c)+\Psi_{322}^d(i,j;c)$

$$E\big(\Psi_{322}^d(j,i;c)\big) = E\Big(b_2^{-2}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}u_i\big[m_d^{l_n}(W_i)-m_d(W_i)\big]\Big)$$
$$= E\Big(b_2^{-2}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}\big[m_d^{l_n}(W_i)-m_d(W_i)\big]E\big[u_i|W_i,X_i,V_i,S_j\big]\Big) = 0.$$

$$E\big(E\big[\Psi_{322}^d(j,i;c)|S_j\big]^2\big)^{1/2} = E\big(E\big[b_2^{-2}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}\big[m_d^{l_n}(W_i)-m_d(W_i)\big]E\big(u_i|W_i,X_i,V_i,S_j\big)|S_j\big]^2\big)^{1/2} = 0.$$

$$E\big(E\big[\Psi_{322}^d(j,i;c)|S_i\big]^2\big)^{1/2} = E\Big(E\Big[b_2^{-2}K_{2ji}^{(1)}(V_d)\phi_i\zeta_{ci}p(V_{di})^{-1}u_i\big[m_d^{l_n}(W_i)-m_d(W_i)\big]\Big|S_i\Big]^2\Big)^{1/2}$$
$$\leq \big[\inf_{V_d\in G_{V_d}}p(V_d)\big]^{-1}\sup_{W\in G_W}|m_d^{l_n}(W)-m_d(W)|b_2^{-1}E\Big(\phi_i^2\zeta_{ci}^2E\big[u_i^2|W_i,X_i,V_i,S_j\big]E\big[b_2^{-1}K_{2ji}^{(1)}(V_d)|S_i\big]^2\Big)^{1/2}$$
$$= O(l_n^{-k}b_2^{-1})E\big(\phi_i^2\zeta_{ci}^2\big)^{1/2} = O(l_n^{-k}b_2^{-1}).$$

$$E\big(\Psi_{322}^d(j,i;c)^2\big)^{1/2} = E\big(b_2^{-4}K_{2ji}^{(1)}(V_d)^2\phi_i^2\zeta_{ci}^2p(V_{di})^{-2}u_i^2\big[m_d^{l_n}(W_i)-m_d(W_i)\big]^2\big)^{1/2}$$
$$= b_2^{-3/2}\big[\inf_{V_d\in G_{V_d}}p(V_d)\big]^{-1}\sup_{W\in G_W}|m_d^{l_n}(W_i)-m_d(W_i)|E\big(\phi_i^2\zeta_{ci}^2E\big[u_i^2|W_i,X_i,V_i,S_j\big]E\big[b_2^{-1}K_{2ji}^{(1)}(V_d)^2|S_i\big]\big)^{1/2}$$
$$= O\big(l_n^{-k}b_2^{-3/2}\big)E\big(\phi_i^2\zeta_{ci}^2\big) = O\big(l_n^{-k}b_2^{-3/2}\big).$$

Now, in summary,

$$E_{322d}(c) = U_{322}^{(2)d}(c) = E\big(\Psi_{322}^d(j,i;c)\big) + O_p(n^{-1/2}E\big(E\big[\Psi_{322}^d(j,i;c)|S_j\big]^2\big)^{1/2})$$
$$+ O_p(n^{-1/2}E\big(E\big[\Psi_{322}^d(j,i;c)|S_i\big]^2\big)^{1/2}) + O_p(n^{-1}E\big(\Psi_{322}^d(j,i;c)^2\big)^{1/2})$$
$$= O_p(n^{-1/2}l_n^{-k}b_2^{-1}) + O_p(n^{-1}l_n^{-k}b_2^{-3/2}) = o_p(n^{-1/2}).$$

Now, note that,

$$E_{32d}(c) = o_p(n^{-1/2}) + E_{321d}(c) + E_{322d}(c) = o_p(n^{-1/2}).$$

Furthermore, $E_{3d} = E_{31d} - E_{32d} + E_{33d} + E_{34d} + E_{35d} = o_p(n^{-1/2})$. Since,

$$E_{4d}(c) = n^{-1}\sum_{i=1}^{n}\zeta_{ci}u_ip(V_{di})^{-1}\phi_i\big(\hat{p}(V_{di}) - E\big[\hat{p}(V_{di})|V_{di}\big]\big),$$

note that the structure of E_{4d} is (exchanging X_d , for V_d) precisely the same as E_{1d}. Thus, in an identical manner as in the case of E_1 it can be shown that, $E_{4d} = o_p(n^{-1/2})$. Since,

$$E_{5d} = n^{-1}\sum_{i=1}^{n}\zeta_{ci}u_i p(V_{di})^{-1}\phi_i\big(E\big[\hat{p}(V_{di})|V_{di}\big] - p(V_{di})\big).$$

note that the structure of E_{5d} is (exchanging X_d, for V_d) precisely the same as E_{2d}. Thus in an identical manner as E_2 it can be shown that, $E_{5d} = o_p(n^{-1/2})$. Since,

$$\begin{aligned}
E_6(c) &= n^{-1}\sum_{i=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i\big[\hat{p}(X_i,\hat{V}_i) - \hat{p}(X_i,V_i)\big]\\
&= n^{-1}\sum_{i=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i[nh_1^D h_2^D]^{-1}\sum_{j=1}^{n}\big[K_{3ji}(X,\hat{V}) - K_{3ji}(X,V)\big]\\
&= [n^2h_1^D h_2^D]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i\big[K_{3ji}(X,\hat{V}) - K_{3ji}(X,V)\big]\\
&= \sum_{d=1}^{D}[n^2h_1^{-D}h_2^{-(D+1)}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i p(V_{di})^{-1}\phi_i D_d K_{3ji}(X,V)\big(\hat{V}_{dj} - V_{dj}\big)\\
&\quad - \sum_{d=1}^{D}[n^2h_1^{-D}h_2^{-(D+1)}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i D_d K_{3ji}(X,V)\big(\hat{V}_{di} - V_{di}\big)\\
&\quad + \sum_{1\le k,d\le D}[2n^2h_1^{-D}h_2^{-(D+2)}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i D_{kd}K_{3ji}(X,V)\prod_{\xi\in\{d,k\}}\big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]^2\\
&\quad + \sum_{1\le m,k,d\le D}[6n^2h_1^D h_2^{D+3}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i D_{mkd}K_{3ji}(X,V)\prod_{\xi\in\{m,d,k\}}\big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]^3\\
&\quad + \sum_{1\le q,m,k,d\le D}[24n^2h_1^D h_2^{D+4}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i D_{qmkd}\tilde{K}_{3ji}(X,V)\prod_{\xi\in\{q,m,d,k\}}\big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]\\
&\equiv E_{61}(c) + E_{62}(c) + E_{63}(c) + E_{64}(c) + E_{65}(c).
\end{aligned}$$

Note that,

$$\begin{aligned}
&E\big[|\phi_i\zeta_{ci}||u_i|p(X_i,V_i)^{-1}|h_1^D h_2^D D_{kd}K_{3ji}(X,V)|\big]\\
&\quad\le \big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-1}E\big[E\big(|h_1^{-D}h_2^{-D}D_{kd}K_{3ji}(X,V)|\big|S_i\big)|\phi_i\zeta_{ci}|E\big(|u_i|\big|W_i,X_i,V_i,S_j\big)\big]\\
&\quad= O\big(1\big)E\big(|\phi_i\zeta_{ci}|\big) = O(1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&E\big[|\phi_i\zeta_{ci}||u_i|p(X_i,V_i)^{-1}|h_1^{-D}h_2^{-D}D_{mkd}K_{3ji}(X,V)|\big]\\
&\quad\le \big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-1}E\big[E\big(|h_1^D h_2^D D_{mkd}K_{3ji}(X,V)|\big|S_i\big)|\phi_i\zeta_{ci}|E\big(|u_i|\big|W_i,X_i,V_i,S_j\big)\big]\\
&\quad= O\big(1\big)E\big(|\phi_i\zeta_{ci}|\big) = O(1).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&E\big[|\phi_i\zeta_{ci}||u_i|p(X_i,V_i)^{-1}|h_1^{-D}h_2^{-(D-4)}D_{qmkd}\tilde{K}_{3ji}(X,V)|\big]\\
&\quad\le \big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-1}E\big[E\big(|h_1^D h_2^{D-4}D_{qmkd}\tilde{K}_{3ji}(X,V)|\big|S_i\big)|\phi_i\zeta_{ci}|E\big(|u_i|\big|W_i,X_i,V_i,S_j\big)\big]\\
&\quad= O\big(1\big)E\big(|\phi_i\zeta_{ci}|\big) = O(1).
\end{aligned}$$

Also note that,

$$
\begin{aligned}
E\big(|\phi_i\zeta_{ci}|^2u_i^2p(X_i,V_i)^{-2}|W_i,V_{di}\big) &\leq \big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-2}E\big[|\phi_i\zeta_{ci}|^2E\big(u_i^2|W_i,V_i,X_i\big)|W_i,V_{di}\big] \\
&= O(1)E\big[|\phi_i\zeta_{ci}|^2|W_i,V_{di}\big] = O(1).
\end{aligned}
$$

A trivial modification of Lemma 6 gives,

$$
\begin{aligned}
\sup_{V_{dj}\in G_{V_d}} \Big|[nh_1^Dh_2^{D+1}]^{-1}\sum_{i=1}^{n} & p(X_i,V_i)^{-1}\phi_i\zeta_{ci}u_iD_dK_{3ji}(X,V) \\
& - E\big[p(X_i,V_i)^{-1}\phi_i\zeta_{ci}u_iD_dK_{3ji}(X,V)\big]\Big| = O_p\left(\left[\frac{\log(n)}{nh_1^Dh_2^{D+2}}\right]^{1/2}\right).
\end{aligned}
$$

$$
\begin{aligned}
&E_{63}(c)+E_{64}(c)+E_{65}(c) \\
&= \sum_{1\leq k,d\leq D}[2n^2h_1^Dh_2^{D+2}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_{kd}K_{3ji}\prod_{\xi\in\{d,k\}}\big[(\hat V_{\xi j}-V_{\xi j})-\big(\hat V_{\xi i}-V_{\xi i}\big)\big]^2 \\
&\quad+\sum_{1\leq m,k,d\leq D}[6n^2h_1^Dh_2^{D+3}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_{mkd}K_{3ji}(X,V)\prod_{\xi\in\{m,d,k\}}\big[(\hat V_{\xi j}-V_{\xi j})-\big(\hat V_{\xi i}-V_{\xi i}\big)\big]^3 \\
&\quad+\sum_{1\leq q,m,k,d\leq D}[24n^2h_1^Dh_2^{D+4}]^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_{qmkd}\tilde K_{3ji}(X,V)\prod_{\xi\in\{q,m,d,k\}}\big[(\hat V_{\xi j}-V_{\xi j})-\big(\hat V_{\xi i}-V_{\xi i}\big)\big]^4 \\
&\leq \max_{1\leq d\leq D}\big[2\sup_{W\in G_W}|\hat m_d^{l_n}(W)-m_d(W)|\big]^2h_2^{-2} \\
&\qquad\times\sum_{1\leq k,d\leq D}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big|\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}[h_1^Dh_2^D]^{-1}D_{kd}K_{3ji}(X,V)\big| \\
&\quad+\max_{1\leq d\leq D}\big[2\sup_{W\in G_W}|\hat m_d^{l_n}(W)-m_d(W)|\big]^3h_2^{-3} \\
&\qquad\times\sum_{1\leq m,k,d\leq D}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big|\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}[h_1^Dh_2^D]^{-1}D_{mkd}K_{3ji}(X,V)\big| \\
&\quad+\max_{1\leq d\leq D}\big[2\sup_{W\in G_W}|\hat m_d^{l_n}(W_i)-m_d(W_i)|\big]^4h_2^{-8} \\
&\qquad\times\sum_{1\leq q,m,k,d\leq D}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\big|\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}[h_1^Dh_2^{D-4}]^{-1}D_{qmkd}\tilde K_{3ji}(X,V)\big| \\
&= O_p\left(\left[\frac{L_n}{h_2}\right]^2\right)+O_p\left(\left[\frac{L_n}{h_2}\right]^3\right)+O_p\left(\left[\frac{L_n}{h_2^2}\right]^4\right)=o_p(n^{-1/2}).
\end{aligned}
$$

by Lemma 3 vi) and vii).

$$
\begin{aligned}
E_{61}(c) &= \sum_{d=1}^{D}[n^2h_1^Dh_2^D]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_dK_{3ji}(X,V)\big(\hat V_{dj}-V_{dj}\big) \\
&= (-1)\sup_{W\in G_W}|\hat m_d^{l_n}(W)-m_d(W)| \\
&\qquad\times\Big\{n^{-1}\sum_{j\neq i}^{n}\sup_{V_{dj}\in G_{V_d}}\Big|[nh_1^Dh_2^{D+1}]^{-1}\sum_{i=1}^{n}\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_dK_{3ji}-E\big[\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_dK_{3ji}(X,V)\big]\Big| \\
&\qquad\quad+[nh_1^Dh_2^{D+1}]^{-1}E\big[\zeta_{ci}u_i\phi_ip(X_i,V_i)^{-1}D_dK_{3ji}(X,V)\big]\Big\} \\
&= O_p(L_n)O_p\left(\left[\frac{\log(n)}{nh_1^Dh_2^{D+2}}\right]^{1/2}\right)=o_p(n^{-1/2}).
\end{aligned}
$$

$$
\begin{aligned}
E_{62d}(c) &= [n^2h_1^Dh_2^{D+1}]^{-1}\sum_{i\neq j}^{n}\sum_{j=1}^{n}\zeta_{ci}u_i\phi_i p(X_i,V_i)^{-1}D_dK_{3ji}(X,V)\big(\hat{m}_d^{l_n}(W_i)-m_d(W_i)\big)\\
&= n^{-1}\sum_{j=1}[nh_1^Dh_2^{D+1}]^{-1}\sum_{i\neq j}^{n}\zeta_{ci}u_i\phi_i p(X_i,V_i)^{-1}D_dK_{3ji}(X,V)\big(\hat{m}_d^{l_n}(W_i)-m_d(W_i)\big)\\
&= n^{-1}\sum_{j=1}^{n}[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\mathbf{u}}_nI_n(-j)\big[\hat{\mathbf{M}}_d^{l_n}-\mathbf{M}_d\big]\\
&\leq n^{-1}\sum_{j=1}^{n}||[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n||_E O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right)\\
&\qquad - n^{-1}\sum_{j=1}^{n}[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\mathbf{u}}_nI_n(-j)\big[\mathbf{M}_d^{l_n}-\mathbf{M}_d\big]\\
&= E_{621d}(c)-E_{622d}(c).
\end{aligned}
$$

$$
E_{621d}(c) = n^{-1}\sum_{j=1}^{n}||[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\boldsymbol{u}}_nI_n(-j)\mathbf{B}_n||_E O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right).
$$

Consider

$$
\begin{aligned}
&E\Big(||[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\mathbf{u}}_nI_n(-j)\mathbf{B}_n||_E^2\Big).\\
&\quad = E\Big(||[nh_1^Dh_2^{D+1}]^{-1}\sum_{i\neq j}\mathbf{B}_n(W_i)'\phi_i\zeta_{ci}p(X_i,V_i)^{-1}u_iD_dK_{3ji}(X,V)||_E^2\Big)\\
&\quad \leq \big[\inf_{X,V\in G_{XV}}p(X,V)\big]^{-2}\\
&\qquad\quad \times[n^2h_1^Dh_2^{D+2}]^{-1}\sum_{i\neq j}E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\phi_i^2\zeta_{ci}^2E\big[u_i^2|W_i,X_i,V_i,S_j\big]E\big[[h_1^Dh_2^D]^{-1}D_dK_{3ji}(X,V)|S_i\big]\Big)\\
&\qquad\quad + \big[\inf_{X,V\in G_{XV}}p(X,V)\big]^{-2}\sum_{i\neq j}\sum_{\substack{g\neq j\\ g\neq i}}E\Big(\mathbf{B}_n(W_i)'\phi_i\zeta_{ci}p(X_i,V_i)^{-1}D_dK_{3ji}(X,V)E\big[u_i|W_i,V_i,X_i,S_{-i}\big]\\
&\qquad\qquad\qquad\qquad \times\mathbf{B}_n(W_g)\phi_g\zeta_{cg}p(X_g,V_g)^{-1}D_dK_{3jg}(X,V)E\big[u_g|W_g,V_g,X_g,S_{-g}\big]\Big)\Big]\\
&\quad = O\big([nh_1^Dh_2^{D+2}]^{-1}\big)E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)E\big(\phi_i^2\zeta_{ci}^2|W_i\big)\Big)\\
&\quad = O\big([nh_1^Dh_2^{D+2}]^{-1}\big)E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\Big) = O\big(l_n[nh_1^Dh_2^{D+2}]^{-1}\big).
\end{aligned}
$$

Consequently, by Markov's Inequality,

$$
\begin{aligned}
E_{621d}(c) &= n^{-1}\sum_{j=1}^{n}||[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\zeta}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\boldsymbol{u}}_nI_n(-j)\mathbf{B}_n||_E O_p\left(\left[\frac{l_n}{n}\right]^{1/2}+l_n^{-k}\right)\\
&= O_p\left(\frac{l_n}{n(h_1^Dh_2^{D+2})^{1/2}}\right) = o_p(n^{-1/2}).
\end{aligned}
$$

by Lemma 3 iii), Now,

$$
\begin{aligned}
E_{622d}(c) &= n^{-1}\sum_{j=1}[nh_1^Dh_2^{D+1}]^{-1}\mathbf{D}_dK_{3j}(X,V)'\boldsymbol{\phi}_n\dot{\boldsymbol{\zeta}}_{cn}\dot{\mathbf{p}}(X,V)^{-1}\dot{\mathbf{u}}_nI_n(-j)\big[\mathbf{M}_d^{l_n}-\mathbf{M}_d\big]\\
&= n^{-2}\sum_{j=1}^{n}\sum_{i\neq j}[h_1^Dh_2^{D+1}]^{-1}D_dK_{3ji}(X,V)\phi_i\zeta_{ci}p(X_i,V_i)^{-1}u_i\big[m_d^{l_n}(W_i)-m_d(W_i)\big]
\end{aligned}
$$

$$= n^{-2}\sum_{j=1}^{n}\sum_{i\neq j}\Psi^d_{622}(j,i;c) \simeq \binom{n}{2}^{-1}\sum_{j=1}^{n}\sum_{j<i}\Gamma^d_{622}(j,i;c) = U^{(2)d}_{622}(c)$$

where $\Gamma^d_{622}(j,i;c) = \Psi^d_{622}(j,i;c) + \Psi^d_{622}(i,j;c)$,

$$\begin{aligned} E\big(\Psi^d_{622}(j,i;c)\big) &= E\Big([h_1^D h_2^{D+1}]^{-1} D_d K_{3ji}(X,V)\phi_i\zeta_{ci}p(X_i,V_i)^{-1}\big[m_d^{l_n}(W_i) - m_d(W_i)\big]E\big(u_i|W_i,X_i,V_i,S_j\big)\Big) \\ &= 0. \end{aligned}$$

$$\begin{aligned} &E\Big(E\big[\Psi^d_{622}(j,i;c)|S_j\big]^2\Big)^{1/2} \\ &\quad = E\Big(E\Big[[h_1^D h_2^{D+1}]^{-1} D_d K_{3ji}(X,V)\phi_i\zeta_{ci}p(X_i,V_i)^{-1}\big[m_d^{l_n}(W_i) - m_d(W_i)\big]E\big(u_i|W_i,X_i,V_i,S_j\big)|S_j\Big]^2\Big)^{1/2} \\ &\quad = 0. \end{aligned}$$

$$\begin{aligned} E\Big(E\big[\Psi^d_{622}(j,i;c)|S_i\big]^2\Big)^{1/2} &= E\Big(\phi_i^2\zeta_{ci}^2 p(X_i,V_i)^{-2}u_i^2\big[m_d^{l_n}(W_i) - m_d(W_i)\big]^2 E\Big[h_1^D h_2^{D+1}\big]^{-1} D_d K_{3ji}(X,V)|S_j\big]^2\Big)^{1/2} \\ &= O(1)\sup_{W\in G_W}|m_d^{l_n}(W) - m_d(W)|\big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-1}E\Big(\phi_i^2\zeta_{ci}^2 E\big[u_i^2|W_i,X_i,V_i\big]\Big)^{1/2} \\ &= O(l_n^{-k})E\Big(\phi_i^2\zeta_{ci}^2\Big)^{1/2} = O(l_n^{-k}). \end{aligned}$$

Integration by parts,

$$\begin{aligned} E\Big(\Psi^d_{622}(j,i;c)^2\Big)^{1/2} &= E\Big([h_1^D h_2^{D+1}]^{-2} D_d K_{3ji}(X,V)^2\phi_i^2\zeta_{ci}^2 p(X_i,V_i)^{-2}u_i^2\big[m_d^{l_n}(W_i) - m_d(W_i)\big]^2\Big) \\ &\leq \sup_{W\in G_W}|m_d^{l_n}(W) - m_d(W)|\big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-1}[h_1^D h_2^{D+2}]^{-1/2} \\ &\qquad \times E\Big(\phi_i^2\zeta_{ci}^2 E\Big[u_i^2|W_i,X_i,V_i,S_j\Big]E\Big[[h_1^D h_2^D]^{-1}D_d K_{3ji}(X,V)^2\big|S_i\Big]\Big)^{1/2} \\ &= O\big(l_n^{-k}[h_1^D h_2^{D+2}]^{-1/2}\big)E\Big(\phi_i^2\zeta_{ci}^2\Big) = O\big(l_n^{-k}[h_1^D h_2^{D+2}]^{-1/2}\big). \end{aligned}$$

Thus in all,

$$\begin{aligned} E^d_{622}(c) = U^{(2)d}_{622}(c) &= E\Big(\Psi^d_{622}(j,i;c)\Big) + O_p\big(n^{-1/2}E\Big(E\big[\Psi^d_{622}(j,i;c)|S_j\big]^2\Big)^{1/2}\big) \\ &\qquad + O_p\big(n^{-1/2}E\Big(E\big[\Psi^d_{622}(j,i;c)|S_i\big]^2\Big)^{1/2}\big) + O_p\big(n^{-1}E\Big(\Psi^d_{622}(j,i;c)^2\Big)^{1/2}\big) \\ &= O_p\big(n^{-1/2}l_n^{-k}\big) + O_p\Big(l_n^{-k}\big[n\big(h_1^D h_2^{D+2}\big)^{1/2}\big]^{-1}\Big) = o_p\big(n^{-1/2}\big), \end{aligned}$$

by Lemma 3 i). Furthermore, $E_{62d}(c) = o_p(n^{-1/2}) + E_{621d}(c) + E_{622d}(c) = o_p(n^{-1/2})$. Consequently, $E_{6d}(c) = E_{61d}(c) + E_{62d}(c) + E_{63d}(c) + E_{64d}(c) + E_{65d}(c) = o_p(n^{-1/2})$. Now,

$$\begin{aligned} E_7(c) &= n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}[h_1^D h_2^D]^{-1}\phi_i\zeta_{ci}u_i p(X_i,V_i)^{-1}\big[K_{3ji}(X,V) - E\big(K_{3ji}(X,V)|X_i,V_i\big)\big] \\ &= n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_7(i,j;c) \simeq \binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma^{(2)}_7(i,j;c) = U^{(2)d}_7(c). \end{aligned}$$

where, $\Gamma^{(2)}_7(i,j;c) = \Psi_7(i,j;c) + \Psi_7(j,i;c)$

$$E\big(\Psi_7(i,j;c)\big) = E\Big([h_1^D h_2^D]^{-1}\phi_i\zeta_{ci}p(X_i,V_i)^{-1}\big[K_{3ji}(X,V) - E\big(K_{3ji}(X,V)|X_i,V_i\big)\big]E\big[u_i|W_i,X_i,V_i,S_j\big]\Big)$$

$$= 0,$$

$$E\Big(E\big[\Psi_7(i,j;c)|S_i\big]^2\Big)^{1/2} = E\Big([h_1^D h_2^D]^{-2}\phi_i^2\zeta_{ci}^2 u_i^2 p(X_i,V_i)^{-2} E\Big[K_{3ji}(X,V) - E\big(K_{3ji}(X,V)|X_i,V_i\big)\Big|S_i\Big]^2\Big)^{1/2}$$
$$= 0,$$

$$E\Big(E\big[\Psi_7(i,j;c)|S_j\big]^2\Big)^{1/2} = E\Big(E\Big[[h_1^D h_2^D]^{-1}\phi_i\zeta_{ci} p(X_i,V_i)^{-1}\big[K_{3ji}(X,V) - E\big(K_{3ji}(X,V)|X_i,V_i\big)\big]E\big[u_i|W_i,X_i,V_i,S_j\big]\Big|S_j\Big]$$
$$= 0,$$

$$\begin{aligned} E\Big(\Psi_7(i,j;c)^2\Big)^{1/2} &= E\Big([h_1^D h_2^D]^{-2}\phi_i^2\zeta_{ci}^2 u_i^2 p(X_i,V_i)^{-2} E\big[K_{3ji}(X,V) - E\big(K_{3ji}(X,V)|X_i,V_i\big)\big]^2\Big)^{1/2} \\ &\le [h_1^D h_2^D]^{-1/2}\big[\inf_{X,V\in G_{XV}} p(X,V)\big]^{-1} E\Big(\phi_i^2\zeta_{ci}^2 E\big[[h_1^D h_2^D]^{-1}K_{3ji}(X,V)^2|S_i\big]E\big[u_i^2|W_i,X_i,V_i,S_j\big]\Big)^{1/2} \\ &= O([h_1^D h_2^D]^{-1/2})E\big(\phi_i^2\zeta_{ci}^2\big) = O([h_1^D h_2^D]^{-1/2}), \end{aligned}$$

$$\begin{aligned} E_7(c) = U_7^{(2)d}(c) &= E\Big(\Psi_7(i,j;c)\Big) + O_p\big(n^{-1/2}E\Big(E\big[\Psi_7(i,j;c)|S_i\big]^2\Big)^{1/2}\big) \\ &\quad + O_p\big(n^{-1/2}E\Big(E\big[\Psi_7(i,j;c)|S_j\big]^2\Big)^{1/2}\big) + O_p\big(n^{-1}E\Big(E\big[\Psi_7(i,j;c)|S_j\big]^2\Big)^{1/2}\big) \\ &= O\big([nh_1^{D/2}h_2^{D/2}]^{-1}\big) = o_p(n^{-1/2}), \end{aligned}$$

$$E_8(c) = n^{-1}\sum_{i=1}^n \zeta_{ci}u_i p(X_i,V_i)^{-1}\phi_i\big(E\big[\hat{p}(X_i,V_i)|X_i,V_i\big] - p(X_i,V_i)\big).$$

In an almost identical manner as E_2 one has, $E_8(c) = O_p(n^{-1/2}(h_1^{\nu_3} + h_2^{\nu_3})) = o_p(n^{-1/2})$. Thus in all combining orders,

$$n^{-1}\sum_{i=1}^n \zeta_i u_i(\hat{\phi}_i - \phi_i) = = o_p(n^{-1/2}).$$

□

Proof of Theorem 1
Under the assumptions A1 - A5 of this paper the proof of uniform convergence of $\hat{p}(X_{di})$ to $p(X_{di})$, M_{2n} is well established in the literature and will not be repeated here, and the proof of uniform convergence of $\hat{p}(\hat{V}_{di})$ to $p(V_{di})$ follows from a simplification of the proof of uniform convergence of $\hat{p}(X_i\hat{V}_i)$ to $p(X_i,V_i)$. Accordingly I will only prove that latter case.

$$\begin{aligned} \hat{p}(X_i,\hat{V}_i) - \hat{p}(X_i,V_i) &= [nh_1^D h_2^D]^{-1}\sum_{j=1}^n \Big(K_3[H^{-1}\{(X_j,\hat{V}_j) - (X_i,\hat{V}_i\}] - K_3[H^{-1}\{(X_j,V_j) - (X_i,V_i\}]\Big) \\ &= [nh_1^D h_2^D]^{-1}\sum_{j=1}^n \big[K_{3ji}(X,\hat{V}) - K_{3ji}(X,V)\big] \\ &= \sum_{d=1}^D [nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^n D_d K_{3ji}(X,V)\big[\hat{V}_{dj} - V_{dj}\big] - \sum_{d=1}^D [nh_1^D h_2^{D+1}]^{-1}\sum_{j=1} D_d K_{3ji}(X,V)\big[\hat{V}_{di} - V_{di}\big] \\ &\quad + \sum_{1\le k,d\le D}[2nh_1^D h_2^{D+2}]^{-1}\sum_{j\ne i} D_{kd}K_{3ji}(X,V)\prod_{\xi\in\{d,k\}}\big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]^2 \end{aligned}$$

$$+\sum_{1\le m,k,d\le D}[6nh_1^D h_2^{D+3}]^{-1}\sum_{j\neq i} D_{mkd}K_{3ji}(X,V)\prod_{\xi\in\{m,d,k\}}\big[(\hat{V}_{\xi j}-V_{\xi j})-\big(\hat{V}_{\xi i}-V_{\xi i}\big)\big]^3$$
$$+\sum_{1\le q,m,k,d\le D}[24nh_1^D h_2^{D+4}]^{-1}\sum_{j\neq i} D_{qmkd}\tilde{K}_{3ji}(X,V)\prod_{\xi\in\{q,m,d,k\}}\big[(\hat{V}_{\xi j}-V_{\xi j})-\big(\hat{V}_{\xi i}-V_{\xi i}\big)\big]^4$$
$$\equiv A_1-A_2+A_3+A_4+A_5.$$

$$\begin{aligned}A_1&=\sum_{d=1}^{D}[nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^{n}D_dK_{3ji}(X,V)\big[\hat{V}_{dj}-V_{dj}\big]\\&=-\sum_{d=1}^{D}[nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^{n}D_dK_{3ji}(X,V)\big[\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big]\equiv-\sum_{d=1}^{D}A_{1d}.\end{aligned}$$

$$\begin{aligned}A_{1d}&=[nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^{n}D_dK_{3ji}\big[\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big]=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\big[\hat{\mathbf{M}}_d^{l_n}-\mathbf{M}_d\big]\\&=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\big[\mathbf{B}_n\hat{\alpha}_d^{l_n}-\mathbf{M}_d\big]=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\big[\mathbf{B}_n\big(\hat{\alpha}_d^{l_n}-\alpha_d^{l_n}+\alpha_d^{l_n}\big)-\mathbf{M}_d\big]\\&=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\big[\mathbf{B}_n\big(\hat{\alpha}_d^{l_n}-\alpha_d^{l_n}\big)\big]+[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\big[\mathbf{B}_n\alpha_d^{l_n}-\mathbf{M}_d\big]\\&=A_{11d}+A_{12d}.\end{aligned}$$

$$\begin{aligned}A_{11d}&=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\big[\mathbf{B}_n\big(\hat{\alpha}_d^{l_n}-\alpha_d^{l_n}\big)\big]\\&=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n\Big[\big(n^{-1}\mathbf{B}_n'\mathbf{B}_n\big)^{-1}n^{-1}\mathbf{B}_n'\big[\mathbf{X}_d-\mathbf{B}_n\alpha_d^{l_n}\big]\Big]\\&=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n\Big[\big(Q_{nBB}^{-1}-Q_{BB}^{-1}+Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d+\mathbf{V}_d-\mathbf{M}_d^{l_n}\big]\Big]\\&=[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_nQ_{BB}^{-1}n^{-1}\mathbf{B}_n'\mathbf{V}_d\\&\quad+[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_nQ_{BB}^{-1}n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]\\&\quad+[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n'\mathbf{V}_d\\&\quad+[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]\\&\equiv A_{111d}+A_{112d}+A_{113d}+A_{114d}.\end{aligned}$$

Note that $||\cdot||_{sp}$ is the matrix norm on the space of matrices of order $l_n\times l_n$ induced by the Euclidean norm on $\mathbb{R}^{l_n}$. Consequently by the Cauchy Schwartz inequality, one has

$$\begin{aligned}\big|A_{111d}\big|&=\big|[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_nQ_{BB}^{-1}n^{-1}\mathbf{B}_n'\mathbf{V}_d\big|\\&\le||[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n||_E\;||Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n'\mathbf{V}_d||_E\\&=O_p(1)O(1)O_p(l_n/\sqrt{n})=O_p(\sqrt{l_n}/\sqrt{n}).\end{aligned}$$

By Lemmas 4 and 5.

$$\begin{aligned}\big|A_{112d}\big|&=\big|[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_nQ_{BB}^{-1}n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]\big|\\&\le||[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n||_E\;||Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]||_E\\&=O_p(1)O(1)O_p(l_n^{-k})=O_p(l_n^{-k}).\end{aligned}$$

By Lemmas 4 and 5.

$$\begin{aligned}\big|A_{113d}\big|&=\big|[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n'\mathbf{V}_d\big|\\&\le||[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_dK_{3i}(X,V)'\mathbf{B}_n||_E\;||Q_{nBB}^{-1}-Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n'\mathbf{V}_d||_E\end{aligned}$$

$$= O_p(1)O_p(l_n/\sqrt{n})O_p(\sqrt{l_n}/\sqrt{n}) = o_p(\sqrt{l_n}/\sqrt{n}).$$

By Lemmas 4 and 5.

$$\begin{aligned}\big|A_{114d}\big| &= \big|[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_d K_{3i}(X,V)'\mathbf{B}_n\big(Q_{nBB}^{-1} - Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d - \mathbf{M}_d^{l_n}\big]\big| \\ &\leq ||[nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_d K_{3i}(X,V)'\mathbf{B}_n||_E\, ||Q_{nBB}^{-1} - Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n'\big[\mathbf{M}_d - \mathbf{M}_d^{l_n}\big]||_E \\ &= O_p(1)O_p(l_n/\sqrt{n})O_p(l_n^{-k}) = o_p(l_n^{-k}).\end{aligned}$$

Consequently,

$$\begin{aligned}A_{11d} &= A_{111d} + A_{112d} + A_{113d} + A_{114d} \\ &= O_p\big(\sqrt{l_n}/\sqrt{n} + l_n^{-k}\big).\end{aligned}$$

Now,

$$\begin{aligned}A_{12d} &= [nh_1^D h_2^{D+1}]^{-1}\mathbf{D}_d K_{3i}(X,V)'\big[\mathbf{B}_n\alpha_d^{l_n} - \mathbf{M}_d\big] \\ &= [nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^n D_d K_{3ji}(X,V)\big[m_d^{l_n}(W_j) - m_d(W_j)\big].\end{aligned}$$

Note that,

$$\begin{aligned}E\big(A_{12d}^2\big) &= E\Big(\Big[[nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^n D_d K_{3ji}(X,V)\big[m_d^{l_n}(W_j) - m_d(W_j)\big]\Big]^2\Big) \\ &= [n^2 h_1^D h_2^{D+2}]^{-1}\sum_{j=1}^n E\Big([h_1^D h_2^D]^{-1} D_d K_{3ji}(X,V)^2\big[m_d^{l_n}(W_j) - m_d(W_j)\big]^2\Big) \\ &\quad + n^{-2}\sum_{j=1}^n\sum_{g\neq j}^n E\Big([h_1^D h_2^{D+1}]^{-1} D_d K_{3ji}(X,V)\big[m_d^{l_n}(W_j) - m_d(W_j)\big] \\ &\qquad\qquad \times [h_1^D h_2^{D+1}]^{-1} D_d K_{3gi}(X,V)\big[m_d^{l_n}(W_g) - m_d(W_g)\big]\Big) \\ &= [n^2 h_1^D h_2^{D+2}]^{-1}\sum_{j=1}^n E\Big(E\Big[[h_1^D h_2^D]^{-1} D_d K_{3ji}(X,V)^2\Big|W_j\Big]\big[m_d^{l_n}(W_j) - m_d(W_j)\big]^2\Big) \\ &\quad + n^{-2}\sum_{j=1}^n\sum_{g\neq j}^n E\Big(E\Big[[h_1^D h_2^{D+1}]^{-1} D_d K_{3ji}(X,V)\Big|V_i, W_j, S_g\Big]\big[m_d^{l_n}(W_j) - m_d(W_j)\big] \\ &\qquad\qquad \times E\Big[[h_1^D h_2^{D+1}]^{-1} D_d K_{3gi}(X,V)\Big|V_i, W_g, S_j\Big]\big[m_d^{l_n}(W_g) - m_d(W_g)\big]\Big) \\ &\leq O_p([nh_1^D h_2^{D+2}]^{-1} + 1)\sup_{W\in G_W}|m_d^{l_n}(W) - m_d(W)|^2 \\ &= O_p\left(\frac{l_n^{-2k}}{nh_1^D h_2^{D+2}} + l_n^{-2k}\right) = O_p(l_n^{-2k})(o(1) + 1).\end{aligned}$$

by Lemma 3 i) and integration by parts. Consequently by Markov's Inequality $A_{12d} = O_p(l_n^{-k})$. In all,

$$A_1 = \sum_{d=1}^D\big(A_{11d} + A_{12d}\big) = O_p\left(\sqrt{\frac{l_n}{n}} + l_n^{-k}\right).$$

$$\begin{aligned}A_{2d} &= [m_d(W_i) - \hat{m}_d^{l_n}(W_i)][nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^n D_d K_{3ji}(X,V) \\ &\leq \sup_{W\in G_W}|m_d(W) - \hat{m}_d^{l_n}(W)|\Big\{\sup_{V_{di}\in G_{V_d}}\Big|[nh_1^D h_2^{D+1}]^{-1}\sum_{j=1}^n\big(D_d K_{3ji}(X,V) - E\big[D_d K_{3ji}(X,V)\big]\big)\Big|\end{aligned}$$

$$+ \sup_{V_{di} \in G_{V_d}} \Big| [nh_1^D h_2^{D+1}]^{-1} \sum_{j=1}^{n} E\big[D_d K_{3ji}(X,V)\big] \Big| \Big\}$$

$$= O_p(L_n) \left\{ O_p\left(\left[\frac{\log(n)}{nh_1^D h_2^{D+2}} \right]^{1/2} \right) + O(1) \right\} = O_p(L_n).$$

By Lemma 3 xvii).

$$\begin{aligned}
A_3 + A_4 + A_5 = &\sum_{1 \le k,d \le D} [2nh_1^D h_2^{D+2}]^{-1} \sum_{j \ne i} D_{kd} K_{3ji}(X,V) \prod_{\xi \in \{d,k\}} \big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]^2 \\
&+ \sum_{1 \le m,k,d \le D} [6nh_1^D h_2^{D+3}]^{-1} \sum_{j \ne i} D_{mkd} K_{3ji}(X,V) \prod_{\xi \in \{m,d,k\}} \big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]^3 \\
&+ \sum_{1 \le q,m,k,d \le D} [24nh_1^D h_2^{D+4}]^{-1} \sum_{j \ne i} D_{qmkd} \tilde{K}_{3ji}(X,V) \prod_{\xi \in \{q,m,d,k\}} \big[(\hat{V}_{\xi j} - V_{\xi j}) - \big(\hat{V}_{\xi i} - V_{\xi i}\big)\big]^4 \\
\le &\; C \max_{1 \le d \le D} \big[2 \sup_{W \in G_W} |m_d^{l_n}(W) - m_d(W)|\big]^2 [nh_2^2]^{-1} \sum_{j \ne i} [h_1^D h_2^D]^{-1} |D_{kd} K_{3ji}(X,V)| \\
&+ C \max_{1 \le d \le D} \big[2 \sup_{W \in G_W} |m_d^{l_n}(W) - m_d(W)|\big]^3 [nh_2^3]^{-1} \sum_{j \ne i} [h_1^D h_2^D]^{-1} |D_{kmd} K_{3ji}(X,V)| \\
&+ C \max_{1 \le d \le D} \big[2 \sup_{W \in G_W} |m_d^{l_n}(W) - m_d(W)|\big]^4 [nh_2^8]^{-1} \sum_{j \ne i} [h_1^D h_2^{D-4}]^{-1} |D_{qkmd} \tilde{K}_{3ji}(X,V)| \\
= &\; O_p\Big(\frac{L_n^2}{h_2^2} + \frac{L_n^3}{h_2^3} + \frac{L_n^4}{h_2^8} \Big) = o_p(n^{-1/2}).
\end{aligned}$$

By Lemma 3 vi) and vii). Now, in summary, noting that each of the preceding result applies uniformly one has,

$$\sup_{X_i, V_i \in G_{XV}} |\hat{p}(X_i, \hat{V}_i) - \hat{p}(X_i, V_i)| = O_p(L_n) + o_p(n^{-1/2}).$$

Consequently under the assumptions A2, and A3, Theorem 1.4 of Li & Racine (2000) states that,

$$\begin{aligned}
\sup_{X_i, V_i \in G_{XV}} |\hat{p}(X_i, \hat{V}_i) - p(X_i, V_i)| &\le \sup_{X_i, V_i \in G_{XV}} |\hat{p}(X_i, \hat{V}_i) - \hat{p}(X_i, V_i)| + \sup_{X_i, V_i \in G_{XV}} |\hat{p}(X_i, V_i) - p(X_i, V_i)| \\
&= O_p(L_n) + O_p(M_{2n}).
\end{aligned}$$

A proof of the uniform rate of convergence of $\hat{\theta}_1^d(X_i, \hat{V}_i)$ to $\theta_1^d(X_i, V_i)$ and $\hat{\theta}_2^d(X_i, \hat{V}_i)$ to $\theta_2^d(X_i, V_i)$ follows from a trivial modification to the proof of the uniform rate of convergence of $\hat{\phi}(X_i, \hat{V}_i)$ to $\phi(X_i, V_i)$. Consequently I only provide a proof of the latter case. Note that by Lemma 9,

$$\begin{aligned}
\hat{g}(X_i, \hat{V}_i) - g(X_i, V_i) &= 2 \sum_{d=1}^{D} g(X_{-di}, V_i)\big[\hat{p}(X_{di}) - p(X_{di})\big] + 2 \sum_{d=1}^{D} g(X_i, V_{-di})\big[\hat{p}(\hat{V}_{di}) - p(V_{di})\big] + O_p(L_n^2) \\
&\le 2 \sum_{d=1}^{D} \sup_{X_{-d}, V \in G_{X_{-d}V}} g(X_{-d}, V) \sup_{X_d \in G_{X_d}} |\hat{p}(X_d) - p(X_d)\big] \\
&\quad + 2 \sum_{d=1}^{D} \sup_{X, V_{-d} \in G_{XV_{-d}}} g(X, V_{-d}) \sup_{V_d \in G_{V_d}} |\hat{p}(\hat{V}_d) - p(V_d)| + o_p(n^{-1/2}) \\
&= O_p(\mathcal{L}_{0n}),
\end{aligned}$$

By previous results and Lemma 3 vi). Now consider,

$$\begin{aligned}
\hat{\phi}(X_i, \hat{V}_i) - \phi(X_i, V_i) &= \hat{p}(X_i, \hat{V}_i)^{-1} \hat{g}(X_i, \hat{V}_i) - p(X_i, V_i)^{-1} g(X_i, V_i) \\
&= [\hat{p}(X_i, \hat{V}_i) p(X_i, V_i)]^{-1} \Big(p(X_i, V_i) \hat{g}(X_i, \hat{V}_i) - \hat{p}(X_i, \hat{V}_i) g(X_i, V_i) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= [\hat{p}(X_i,\hat{V}_i)p(X_i,V_i)]^{-1}\Big(p(X_i,V_i)\big[\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big]+g(X_i,V_i)\big[p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big]\Big)\\
&= \hat{p}(X_i,\hat{V}_i)^{-1}\big[\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big]+[\hat{p}(X_i,\hat{V}_i)p(X_i,V_i)]^{-1}g(X_i,V_i)\big[p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big]\\
&= \hat{p}(X_i,\hat{V}_i)^{-1}\big[\big(\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big)+\phi_i\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big]\\
&= \big[p(X_i,V_i)^{-1}+[\hat{p}(X_i,\hat{V}_i)p(X_i,V_i)]^{-1}\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big]\\
&\qquad\times\big[\big(\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big)+\phi_i\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big]\\
&\leq \Big[p(X_i,V_i)^{-1}+\big(p(X_i,V_i)^2+p(X_i,V_i)\big[\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big]\big)^{-1}\big[\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big]\Big]\\
&\qquad\times\big[\big(\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big)+\phi_i\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big]\\
&\equiv \big[A_1+A_2\big]\big[\big(\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big)+\phi_i\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big].
\end{aligned}
$$

Where,

$$
\begin{aligned}
&A_1 = p(X_i,V_i)^{-1},\\
&A_2 = \big(p(X_i,V_i)^2+p(X_i,V_i)\big[\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big]\big)^{-1}\big[\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big].
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&A_2\big[\big(\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big)+\phi_i\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big]\\
&\quad\leq |A_2|\Big(\sup_{X,V\in G_{XV}}|\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)|+\sup_{X,V\in G_{XV}}\phi(X,V)\sup_{X,V\in G_{XV}}|p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)|\Big)\\
&\quad= \big(p(X_i,V_i)^2+p(X_i,V_i)\big[\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big]\big)^{-1}\big[\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big]O_p(\mathcal{L}_{0n})\\
&\quad\leq \Big(\inf_{X,V\in G_{XV}}p(X_i,V_i)^2+\inf_{X,V\in G_{XV}}p(X_i,V_i)o_p(1)\Big)^{-1}\sup_{X,V\in G_{XV}}\big|\hat{p}(X_i,\hat{V}_i)-p(X_i,V_i)\big|O_p(\mathcal{L}_{0n})\\
&\quad= O_p(\mathcal{L}_{0n}^2)=o_p(n^{-1/2}).
\end{aligned}
$$

By Assumption 3, and Lemma 3 xxvi). Also,

$$
\begin{aligned}
&A_1\big[\big(\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)\big)+\phi_i\big(p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)\big)\big]\\
&\quad\leq |A_1|\Big(\sup_{X,V\in G_{XV}}|\hat{g}(X_i,\hat{V}_i)-g(X_i,V_i)|+\sup_{X,V\in G_{XV}}p(X,V)\sup_{X,V\in G_{XV}}|p(X_i,V_i)-\hat{p}(X_i,\hat{V}_i)|\Big)\\
&\quad\leq \Big[\inf_{X,V\in G_{XV}}p(X,V)\Big]^{-1}O_p(\mathcal{L}_{0n})=O(1)O_p(\mathcal{L}_{0n}).
\end{aligned}
$$

By Assumption 3. Hence,

$$
\hat{\phi}(X_i,\hat{V}_i)-\phi(X_i,V_i)=O_p(\mathcal{L}_{0n})+o_p(n^{-1/2}).
$$

□

Proof of Theorem 2 : Let A be one of the component random variables in $[\,Y,X']'$ Note that,

$$
E\big(K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big)=E\big(K_{2ji}^{(1)}(V_d)p(V_{dj})^{-1}E\big[\phi_jA_j|V_{dj},S_i\big]\big)=E\big(E\big[K_{2ji}^{(1)}(V_d)p(V_{dj})^{-1}H_2^d(A_j)\big|S_i\big]\big).
$$

Integration by parts gives,

$$
\begin{aligned}
&E\big[K_{2ji}^{(1)}(V_d)p(V_{dj})^{-1}H_2^d(A_j)\big|S_i\big]=\int K_{2ji}^{(1)}(V_d)p(V_{dj})^{-1}H_2^d(A_j)p(V_{dj})dV_{dj}\\
&\quad= \Big[K_2(b_2^{-1}[V_{dj}-V_{di}])E(\phi_jA_j|V_{dj})\Big]_{-\infty}^{\infty}-b_2\int K_{2ji}(V_d)H_2^{(1)d}(A_j)dV_{dj}\\
&\quad\leq \sup_{V_d\in G_{V_d}}|E\big(\phi A|V_d)|\Big[\lim_{\gamma\to\infty}K_2(\,\gamma\,)-\lim_{\gamma\to-\infty}K_2(\,\gamma\,)\Big]+\sup_{V_d\in G_{V_d}}|E^{(1)}\big(\phi A|V_d)|b_2^2\int|K_2(\gamma)|d\gamma\quad=O(b_2^2).
\end{aligned}
$$

By assumptions A2 and A4. Thus,

$$E\big(K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big) = E\Big(E\big[K_{2ji}^{(1)}(V_d)p(V_{dj})^{-1}H_2^d(A_j)\big|S_i\big]\Big) = O(b_2^2).$$

$$\begin{aligned}
\sup_{X_{di}\in G_{X_d}} & \Big|[(n-1)b_1]^{-1}\sum_{j\neq i}\big(\hat\theta_{1j}^d K_{1ji}(X_d)A_j - \theta_{1j}^d K_{1ji}(X_d)A_j\big)\Big| \\
&\le \sup_{X,V\in G_{XV}} [(n-1)b_1]^{-1}\sum_{j\neq i}\big|K_{1ji}(X_d)\big|\big|\hat\theta_{1i}^d - \theta_{1i}^d\big|\big|A_j\big| \\
&\le \sup_{X,V\in G_{XV}} \big|\hat\theta_1^d(X,\hat V) - \theta_1^d(X,\hat V)\big| \sup_{X_{di}\in G_{X_d}} [(n-1)b_1]^{-1}\sum_{j\neq i}\big|K_{1ji}(X_d)\big|\big|A_j\big|. \\
&= O_p(\mathcal{L}_{0n}).
\end{aligned}$$

By Theorem 1. Furthermore, by Theorem 2.6 in Li and Racine (2007), under the assumptions of this paper,

$$\sup_{X_{di}\in G_{X_d}} \Big|[(n-1)b_1]^{-1}\sum_{j\neq i}\theta_{1j}^d K_{1ji}(X_d)A_j - E\big[\phi_i A_i|X_{di}\big]\Big| = O_p\left(\left[\frac{\log(n)}{nb_1}\right]^{1/2} + b_1^{\nu_1}\right) = O_p(N_{1n}).$$

Consequently, one has,

$$\begin{aligned}
\sup_{X_{di}\in G_{X_d}} \big|\hat H_1^d(A_i) - H_1^d(A_i)\big| &= \sup_{X_{di}\in G_{X_d}} \Big|[(n-1)b_1]^{-1}\sum_{j\neq i}\hat\theta_{1j}^d K_{1ji}(X_d)A_j - E\big[\phi_i A_i|X_{di}\big]\Big| \\
&= O_p\big(\mathcal{L}_{0n} + N_{1n}\big) = O_p(L_n + M_n + N_{1n}) \\
&= O_p(\mathcal{L}_{1n}).
\end{aligned}$$

Now,

$$\begin{aligned}
[(n-1)b_2]^{-1} & \sum_{j\neq i} K_{2ji}(\hat V_d)\hat\theta_{2i}^d A_j - [(n-1)b_2]^{-1}\sum_{j\neq i} K_{2ji}(V_d)\theta_{2i}^d A_j \\
&\le [(n-1)b_2]^{-1}\sum_{j\neq i}\big[K_{2ji}(\hat V_d)\hat\theta_{2j}^d A_j - K_{2ji}(\hat V_d)\theta_{2j}^d A_j + K_{2ji}(\hat V_d)\theta_{2j}^d A_j - K_{2ji}(V_d)\theta_{2i}^d A_j\big] \\
&\le [(n-1)b_2]^{-1}\sum_{j\neq i} K_{2ji}(\hat V_d)\big(\hat\theta_{2j}^d - \theta_{2j}^d\big)A_j + [(n-1)b_2]^{-1}\sum_{j\neq i}\big(K_{2ji}(\hat V_d) - K_{2ji}(V_d)\big)\theta_{2j}^d A_j \\
&= [(n-1)b_2]^{-1}\sum_{j\neq i} K_{2ji}(V_d)\big(\hat\theta_{2j}^d - \theta_{2j}^d\big)A_j \\
&\quad + [(n-1)b_2]^{-1}\sum_{j\neq i}\big(K_{2ji}(\hat V_d) - K_{2ji}(V_d)\big)\big(\hat\theta_{2j}^d - \theta_{2j}^d\big)A_j \\
&\quad + [(n-1)b_2]^{-1}\sum_{j\neq i}\big(K_{2ji}(\hat V_d) - K_{2ji}(V_d)\big)\theta_{2j}^d A_j \\
&\equiv B_1 + B_2 + B_3.
\end{aligned}$$

$$\begin{aligned}
B_1 &= [(n-1)b_2]^{-1}\sum_{j\neq i} K_{2ji}(V_d)\big(\hat\theta_{2j}^d - \theta_{2j}^d\big)A_j \\
&\le \sup_{X,V\in G_{XV}} |\hat\theta_2^d(X,\hat V) - \theta_2^d(X,V)|[(n-1)b_2]^{-1}\sum_{j\neq i}|K_{2ji}(V_d)||A_j| = O_p(\mathcal{L}_{0n}).
\end{aligned}$$

By Taylor expansion one has

$$\big|B_2\big| = \Big|[(n-1)b_2]^{-1}\sum_{j\neq i}\big(K_{2ji}(\hat V_d) - K_{2ji}(V_d)\big)\big(\hat\theta_{2j}^d - \theta_{2j}^d\big)A_j\Big|$$

$$
\begin{aligned}
&\le \sup_{X,V\in G_{XV}} |\hat{\theta}_2^d(X,\hat{V}) - \theta_2^d(X,V)|[(n-1)b_2]^{-1}\sum_{j\neq i}|K_{2ji}(\hat{V}_d) - K_{2ji}(V_d)||A_j| \\
&= O_p(\mathcal{L}_{0n})\Big\{[(n-1)b_2^2]^{-1}\sum_{j\neq i}\big|K_{2ji}^{(1)}(V_d)\big|\big|A_j\big|\big|\big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big) - \big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big)\big| \\
&\qquad + [2(n-1)b_2^3]^{-1}\sum_{j\neq i}\big|K_{2ji}^{(2)}(V_d)\big|\big|A_j\big|\big|\big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big) - \big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big)\big|^2 \\
&\qquad + [6(n-1)b_2^4]^{-1}\sum_{j\neq i}\big|K_{2ji}^{(3)}(V_d)\big|\big|A_j\big|\big|\big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big) - \big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big)\big|^3 \\
&\qquad + [24(n-1)b_2^5]^{-1}\sum_{j\neq i}\big|\tilde{K}_{2ji}^{(4)}(V_d)\big|\big|A_j\big|\big|\big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big) - \big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big)\big|^4\Big\} \\
&= O_p(\mathcal{L}_{0n})\Big\{\sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W) - m_d(W)\big|[(n-1)b_2]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\big|A_j\big| \\
&\qquad + \sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W) - m_d(W)\big|^2[(n-1)b_2^2]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|\big|A_j\big| \\
&\qquad + \sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W) - m_d(W)\big|^3[(n-1)b_2^3]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|\big|A_j\big| \\
&\qquad + \sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W) - m_d(W)\big|^4[(n-1)b_2^5]^{-1}\sum_{j\neq i}\big|\tilde{K}_{2ji}^{(4)}(V_d)\big|\big|A_j\big|\Big\} \\
&= O_p(\mathcal{L}_{0n})\Big\{O_p\big(L_n b_2^{-1}\big)\big|[(n-1)]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\big|A_j\big| \\
&\qquad + O_p\big(L_n^2 b_2^{-2}\big)\big|[(n-1)]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|\big|A_j\big| \\
&\qquad + O_p\big(L_n^3 b_2^{-3}\big)\big|[(n-1)]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|\big|A_j\big| \\
&\qquad + O_p\big(L_n^4 b_2^{-5}\big)\big|\sup_{\gamma\in\mathbb{R}}\big|K_2^{(4)}(\ \gamma\ )\big|[(n-1)]^{-1}\sum_{j\neq i}\big|A_j\big|\Big\} \\
&= O_p(\mathcal{L}_{0n})\Big\{O_p\big(L_n b_2^{-1}\big) + O_p\big(L_n^2 b_2^{-2}\big) + O_p\big(L_n^3 b_2^{-3}\big) + O_p\big(L_n^4 b_2^{-5}\big)\Big\} = o_p(\mathcal{L}_{0n}).
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= [(n-1)b_2]^{-1}\sum_{j\neq i}\big(K_{2ji}(\hat{V}_d) - K_{2ji}(V_d)\big)\theta_{2j}^d A_j \\
&= (-1)[(n-1)b_2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big) \\
&\quad + [(n-1)b_2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big) \\
&\quad + [(n-1)b_2^2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(2)}(V_d)A_j\theta_{2j}^d\big[\big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big) - \big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big)\big]^2 \\
&\quad + [(n-1)b_3^2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(3)}(V_d)A_j\theta_{2j}^d\big[\big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big) - \big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big)\big]^3 \\
&\quad + [(n-1)b_5^2]^{-1}\sum_{j\neq i} \tilde{K}_{2ji}^{(4)}(V_d)A_j\theta_{2j}^d\big[\big(\hat{m}_d^{l_n}(W_i) - m_d(W_i)\big) - \big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big)\big]^4 \\
&\equiv -B_{31} + B_{32} + B_{33} + B_{34} + B_{35}.
\end{aligned}
$$

$$
\begin{aligned}
B_{31} &= [(n-1)b_2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big(\hat{m}_d^{l_n}(W_j) - m_d(W_j)\big) \\
&= [(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^d\dot{\mathbf{A}}_n I_n(-i)\big[\hat{\mathbf{M}}_d^{l_n} - \mathbf{M}_d\big].
\end{aligned}
$$

Now as in the proof of Theorem 1, and given the results of Lemma 4,

$$\begin{aligned}
\big|B_{31}\big| &= \Big|[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n Q_{BB}^{-1}n^{-1}\mathbf{B}_n\mathbf{V}_d \\
&\quad + [(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n\mathbf{V}_d \\
&\quad + [(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n Q_{BB}^{-1}n^{-1}\mathbf{B}_n\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big] \\
&\quad + [(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]\Big| \\
&\leq ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||Q_{BB}^{-1}n^{-1}\mathbf{B}_n\mathbf{V}_d||_E \\
&\quad + ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n\mathbf{V}_d||_E \\
&\quad + ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||Q_{BB}^{-1}n^{-1}\mathbf{B}_n\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]||_E \\
&\quad + ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||\big(Q_{nBB}^{-1}-Q_{BB}^{-1}\big)n^{-1}\mathbf{B}_n\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]||_E \\
&\leq ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n\mathbf{V}_d||_E \\
&\quad + ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||Q_{nBB}^{-1}-Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n\mathbf{V}_d||_E \\
&\quad + ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]||_E \\
&\quad + ||[(n-1)b_2^2]^{-1}\mathbf{K}_{2i}^{(1)}(V_d)'\dot{\boldsymbol{\Theta}}_{2n}^{d}\dot{\mathbf{A}}_n I_n(-i)\mathbf{B}_n||_E||Q_{nBB}^{-1}-Q_{BB}^{-1}||_{sp}||n^{-1}\mathbf{B}_n\big[\mathbf{M}_d-\mathbf{M}_d^{l_n}\big]||_E \\
&= O_p(b_2^{-1})\left[O(1)O_p\left(\sqrt{\frac{l_n}{n}}\right)+O_p\left(\frac{l_n}{\sqrt{n}}\right)O_p\left(\sqrt{\frac{l_n}{n}}\right)+O(1)O_p\left(l_n^{-k}\right)+O_p\left(\frac{l_n}{\sqrt{n}}\right)O_p\left(l_n^{-k}\right)\right] \\
&= O_p\left(b_2^{-1}\left[\sqrt{\frac{l_n}{n}}+l_n^{-k}\right]\right).
\end{aligned}$$

By assumption A1.

$$\begin{aligned}
B_{32} &= [(n-1)b_2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big(\hat{m}_d^{l_n}(W_i)-m_d(W_i)\big) \\
&\leq \sup_{W\in G_W}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|\big|[(n-1)b_2]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big| \\
&\leq O_p(L_n)\Big\{\sup_{V_d\in G_{V_d}}\Big|[(n-1)b_2]^{-1}\sum_{j\neq i}\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d-E\big(b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big)\Big]\Big| \\
&\qquad + \Big|\sup_{V_d\in G_{V_d}}[(n-1)b_2]^{-1}\sum_{j\neq i}E\big(b_2^{-1}K_{2ji}^{(1)}(V_d)A_j\theta_{2j}^d\big)\Big|.
\end{aligned}$$

From the preliminary portion of this proof, and a combination of Assumption A3 and the results of Lemma 6 one obtains,

$$B_{32} = O_p(L_n)\left(O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right)+O(1)\right) = O_p(L_n).$$

$$\begin{aligned}
B_{33}+B_{34}+B_{35} &\leq 2\sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|^2\sup_{X,V\in G_{XV}}\theta_2^d(X,V)[(n-1)b_2^2]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|\big|A_j\big| \\
&\quad + 8/6\sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|^3\sup_{X,V\in G_{XV}}\theta_2^d(X,V)[(n-1)b_2^3]^{-1}\sum_{j\neq i}^{n}\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|\big|A_j\big| \\
&\quad + 16/24\sup_{X,V\in G_{XV}}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|^4\sup_{X,V\in G_{XV}}\theta_2^d(X,V)\sup_{\gamma\in\mathbb{R}}|K_2^{(4)}|(\ \gamma\ )[(n-1)b_2^5]^{-1}\sum_{j\neq i}|A_j|
\end{aligned}$$

$$\leq O_p(L_n^2 b_2^{-2}) + O_p(L_n^3 b_2^{-3}) + O_p(L_n^4 b_2^{-5}) = o_p(n^{-1/2}),$$

by Lemma 3 vi) and vii). In all,

$$\begin{aligned} B_3 &= -B_{31} + B_{32} + B_{33} + B_{34} + B_{35} \\ &= O_p\left(b_2^{-1}\left[\sqrt{\frac{l_n}{n}} + l_n^{-k}\right]\right) + O_p(L_n) + o_p(n^{1/2}) \\ &= O_p\left(L_n + b_2^{-1}\left[\sqrt{\frac{l_n}{n}} + l_n^{-k}\right]\right). \end{aligned}$$

Also,

$$\begin{aligned} &[(n-1)b_2]^{-1}\sum_{j\neq i} K_{2ji}(\hat{V}_d)\hat{\theta}_{2i}^d A_j - [(n-1)b_2]^{-1}\sum_{j\neq i} K_{2ji}(V_d)\theta_{2i}^d A_j \\ &= B_1 + B_2 + B_3 \\ &= O_p\left(L_n + \mathcal{L}_{0n} + b_2^{-1}\left[\sqrt{\frac{l_n}{n}} + l_n^{-k}\right]\right) = O_p\left(L_n + M_n + b_2^{-1}\left[\sqrt{\frac{l_n}{n}} + l_n^{-k}\right]\right). \end{aligned}$$

Furthermore, by Theorem 2.6 in Li and Racine (2007), and under the assumptions A2 and A3,

$$\sup_{V_{di}\in G_{V_d}}\left|[(n-1)b_2]^{-1}\sum_{j\neq i}\left(\theta_{2j}^d K_{2ji}(V_d)A_j - E\big[\phi_i A_i|V_{di}\big]\right)\right| = O_p\left(\left[\frac{\log(n)}{nb_2}\right]^{1/2} + b_2^{\nu_2}\right) = O_p(N_{2n}).$$

Consequently, one has,

$$\begin{aligned} \sup_{V_{di}\in G_{V_d}}\left|\hat{H}_2^d(A_i) - H_2^d(A_i)\right| &= \sup_{V_{di}\in G_{V_d}}\left|[(n-1)b_2]^{-1}\sum_{j\neq i}\left(\hat{\theta}_{2j}^d K_{2ji}(V_d)A_j - E\big[\phi_i A_i|V_{di}\big]\right)\right| \\ &= O_p\left(L_n + M_n + b_2^{-1}\left[\sqrt{\frac{l_n}{n}} + l_n^{-k}\right] + N_{2n}\right) = O_p\big(\mathcal{L}_{2n}\big). \end{aligned}$$

$$\begin{aligned} \hat{\mu}_A - \mu_A &= n^{-1}\sum_{j=1}\left(\hat{\phi}_j A_j - E\big[\phi_j A_j\big]\right) \\ &\leq \sup_{X,V\in G_{XV}}\left|\hat{\phi}(X,\hat{V}) - \phi(X,V)\right| n^{-1}\sum_{j=1}^{n}|A_j| + n^{-1}\sum_{j=1}^{n}\left(\phi_j A_j - E\big[\phi_j A_j\big]\right) \\ &= O_p(\mathcal{L}_{0n}) + O_p(n^{-1/2}) = O_p(\mathcal{L}_{0n}), \end{aligned}$$

by Markov's Inequality.

$$\begin{aligned} \sup_{X,V\in G_{XV}}|\hat{H}^*(A_j) - H^*(A_j)| &\leq \sum_{d=1}^{D}\sup_{X_d\in G_{X_d}}|\hat{H}_1^d(A_j) - H_1^d(A_j)| \\ &\quad + \sum_{d=1}^{D}\sup_{V_d\in G_{V_d}}|\hat{H}_2^d(A_j) - H_2^d(A_j)| + (2D-1)[\hat{\mu}_A - \mu_A] \\ &= O_p\big(\mathcal{L}_{1n} + \mathcal{L}_{2n}\big) = O_p(\mathcal{L}_n). \end{aligned}$$

□

Proof of Theorem 3:

$$\begin{aligned} n^{1/2}\big(\hat{\beta}_1 - \beta_1\big) &= \big[n^{-1}\hat{\boldsymbol{\zeta}}_n'\hat{\boldsymbol{\phi}}_n\hat{\zeta}_n\big]^{-1}\sqrt{n}\big(n^{-1}\hat{\boldsymbol{\zeta}}_n'\hat{\boldsymbol{\phi}}_n\big[\mathbf{Y}_n - \hat{\mathbf{H}}_n^*(Y) - \big(\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big)\beta_1\big]\big) \\ &= A^{-1}\sqrt{n}B. \end{aligned}$$

where

$$A = n^{-1}\big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big],$$
$$B = n^{-1}\big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathbf{Y}_n - \hat{\mathbf{H}}_n^*(Y) - \big(\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big)\beta_1\big].$$

Note that,

$$\begin{aligned}
A &= n^{-1}\big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big] \\
&= n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big] + n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big] \\
&\quad + n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big] + n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big] \\
&\quad + n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big] + n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big] \\
&\quad + n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big] + n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big] \\
&\equiv A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8.
\end{aligned}$$

First, recall that,

$$\Sigma_0 = E\Big([Z - H^*(Z)]'\phi([Z - H^*(Z)]\Big).$$

Let $c, m \in \{1, 2, \ldots, p\}$,

$$A_1(c, m) = n^{-1}\sum_{i=1}^{n}[Z_{ci} - H^*(Z_{ci})]\phi_i[Z_{mi} - H^*(Z_{mi})] = n^{-1}\sum_{i=1}^{n}\zeta_{ci}\phi_i\zeta_{mi}.$$

Define,

$$D_1(c, m) = A_1(c, m) - \Sigma_0(c, m) = n^{-1}\sum_{i=1}^{n}\big[\zeta_{ci}\phi_i\zeta_{mi} - E\big(\zeta_c\phi\zeta_m\big)\big].$$

and $D_1(c, :) = [D_1(c, 1) \ \ D_1(c, 2) \ \ \cdots \ \ D_1(c, p)]'$ so that,

$$D_1(c, :)'D_1(c, :) = \sum_{m=1}^{p}\Big(n^{-1}\sum_{i=1}^{n}\big[\zeta_{ci}\phi_i\zeta_{mi} - E\big(\zeta_c\phi\zeta_m\big)\big]\Big)^2.$$

Since for all $c, m \in \{1, 2, \ldots, p\}$, $E\big(\zeta_c\phi\zeta_m\big) = O(1)$ one has,

$$\begin{aligned}
||A_1 - \Sigma_0||^2 = ||D_1||^2 = \text{trace}\big(D_1D_1'\big) &= \sum_{c=1}^{p}D_1(c, :)'D_1(c, :) \\
&= \sum_{c=1}^{p}\sum_{m=1}^{p}\Big(n^{-1}\sum_{i=1}^{n}\big[\zeta_{ci}\phi_i\zeta_{mi} - E\big(\zeta_c\phi\zeta_m\big)\big]\Big)^2 \\
&= \sum_{c=1}^{p}\Big(n^{-1}\sum_{i=1}^{n}\big[\zeta_{ci}^2\phi_i - E\big(\zeta_c^2\phi\big)\big]\Big)^2 + \sum_{c=1}^{p}\sum_{m\neq c}\Big(n^{-1}\sum_{i=1}^{n}\big[\zeta_{ci}\phi_i\zeta_{mi} - E\big(\zeta_c\phi\zeta_m\big)\big]\Big)^2 \\
&= \sum_{c=1}^{p}o_p(1)^2 + \sum_{c=1}^{p}\sum_{m\neq c}o_p(1)^2 = o_p(1).
\end{aligned}$$

Let $c, m \in \{1, 2, \ldots, p\}$ and define,

$$A_2(c, m) = n^{-1}\sum_{i=1}^{n}\zeta_{ci}\phi_i\big[H^*(Z_{mi}) - \hat{H}^*(Z_{mi})\big].$$

Also let $A_2(c, :) = [A_2(c, 1) \ \ A_2(c, 2) \ \ \cdots \ \ A_2(c, p)]'$ so that by assumption A5, Theorem 2, and Markov's inequality,

$$A_2(c, :)'A_2(c, :) = \sum_{m=1}^{p}\Big(n^{-1}\sum_{i=1}^{n}\zeta_{ci}\phi_i\big[H^*(Z_{mi}) - \hat{H}^*(Z_{mi})\big]\Big)^2$$

$$\leq \sum_{m=1}^{p} \sup_{X,V \in G_{XV}} \left[H^*(Z_m) - \hat{H}^*(Z_m)\right]^2 \Big(n^{-1} \sum_{i=1}^{n} |\zeta_{ci}\phi_i|\Big)^2$$
$$= O_p(\mathcal{L}_n^2) O_p(1) = o_p(1).$$

Consequently, $||A_2|| = \text{trace}(A_2 A_2')^{1/2} = \Big[\sum_{c=1}^{p} A_2(c,:)' A_2(c,:)\Big]^{1/2} = o_p(1)$. Let $c, m \in \{1, 2, \ldots, p\}$ so that,

$$A_3(c,m) = n^{-1} \sum_{i=1}^{n} \zeta_{ci}(\hat{\phi}_i - \phi_i)\zeta_{mi}.$$

Also let $A_3(c,:) = [A_3(c,1) \;\; A_3(c,2) \;\; \cdots \;\; A_3(c,p)]'$ so that by assumption A5, Theorem 2, and Markov's Inequality,

$$A_3(c,:)' A_3(c,:) = \sum_{m=1}^{p} \Big(n^{-1} \sum_{i=1}^{n} \zeta_{ci}(\hat{\phi}_i - \phi_i)\zeta_{mi}\Big)^2$$
$$\leq \sum_{m=1}^{p} \sup_{X,V \in G_{XV}} |\hat{\phi}(X, \hat{V}) - \phi(X,V)|^2 \Big(n^{-1} \sum_{i=1}^{n} |\zeta_{ci}\zeta_{mi}|\Big)^2$$
$$= O_p(\mathcal{L}_n^2) O_p(1) = o_p(1).$$

Consequently, $||A_3|| = \text{trace}(A_3 A_3')^{1/2} = \Big[\sum_{c=1}^{p} A_3(c,:)' A_3(c,:)\Big]^{1/2} = o_p(1)$. Let $c, m \in \{1, 2, \ldots, p\}$ so that,

$$A_4(c,m) = n^{-1} \sum_{i=1}^{n} \zeta_{ci}(\hat{\phi}_i - \phi_i)[H^*(Z_{mi}) - \hat{H}^*(Z_{mi})]$$

Also let $A_4(c,:) = [A_4(c,1) \;\; A_4(c,2) \;\; \cdots \;\; A_4(c,p)]'$ so that by assumption A5, Theorem 2, and Markov's Inequality,

$$A_4(c,:)' A_4(c,:) = \sum_{m=1}^{p} \Big(n^{-1} \sum_{i=1}^{n} \zeta_{ci}(\hat{\phi}_i - \phi_i)[H^*(Z_{mi}) - \hat{H}^*(Z_{mi})]\Big)^2$$
$$\leq \sum_{m=1}^{p} \sup_{X,V \in G_{XV}} |\hat{\phi}(X,V) - \phi(X,V)|^2 \sup_{X,V \in G_{XV}} |H^*(Z_{mi}) - \hat{H}^*(Z_{mi})|^2 \Big(n^{-1} \sum_{i=1}^{n} |\zeta_{ci}|\Big)$$
$$= O_p(\mathcal{L}_{0n}^2) O_p(\mathcal{L}_n^2) O_p(1) = o_p(1)$$

Consequently, $||A_4|| = \text{trace}(A_4 A_4')^{1/2} = \Big[\sum_{c=1}^{p} A_4(c,:)' A_4(c,:)\Big]^{1/2} = o_p(1)$.

$$A_5 = n^{-1} \big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]' \boldsymbol{\phi}_n \big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big].$$

Note the proof of the order of A_5 is, mutatis mutandis, practically identical to the proof of the order of A_2, thus the arguments are not repeated here. Consequently one can conclude that, $||A_5|| = o_p(1)$.
Let $c, m \in \{1, 2, \ldots, p\}$ so that, $A_6(c,m) = n^{-1} \sum_{i=1}^{n} [H^*(Z_{ci}) - \hat{H}^*(Z_{ci})]\phi_i [H^*(Z_{mi}) - \hat{H}^*(Z_{mi})]$. Also let $A_6(c,:) = [A_6(c,1) \;\; A_6(c,2) \;\; \cdots \;\; A_6(c,p)]'$ so that by assumption A5, Theorem 2, and Markov's Inequality,

$$A_6(c,:)' A_6(c,:) = \sum_{m=1}^{p} \Big(n^{-1} \sum_{i=1}^{n} [H^*(Z_{ci}) - \hat{H}^*(Z_{ci})]\phi_i [H^*(Z_{mi}) - \hat{H}^*(Z_{mi})]\Big)^2$$
$$\leq p \max_{1 \leq a \leq p} \sup_{X,V \in G_{XV}} |H^*(Z_a) - \hat{H}^*(Z_a)|^4 n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\phi_i||\phi_j|$$
$$= O_p(\mathcal{L}_n^4) O(1) = o_p(1).$$

Consequently, $||A_6|| = \text{trace}(A_6 A_6')^{1/2} = \Big[\sum_{c=1}^p A_6(c,:)' A_6(c,:) \Big]^{1/2} = o_p(1)$.

$$A_7 = n^{-1} \big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z) \big]' \big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n \big) \big[\mathbf{Z}_n - \mathbf{H}_n^*(Z) \big].$$

The proof of the order of A_7 is, mutatis mutandis, practically identical to the proof of the order of A_4, thus the arguments are not repeated here. Consequently one can conclude that, $||A_7|| = o_p(1)$. Let $c, m \in \{1, 2, \ldots, p\}$ so that,

$$A_8(c,m) = n^{-1} \sum_{i=1}^n [H^*(Z_{ci}) - \hat{H}^*(Z_{ci})] \big(\hat{\phi}_i - \phi_i \big) [H^*(Z_{mi}) - \hat{H}^*(Z_{mi})].$$

Also, let $A_8(c,:) = [A_8(c,1) \;\; A_8(c,2) \;\; \cdots \;\; A_8(c,p)]'$ so that by assumption A5, Theorem 2, and Markov's Inequality,

$$\begin{aligned} A_8(c,:)' A_8(c,:) &= \sum_{m=1}^p \Big(n^{-1} \sum_{i=1}^n [H^*(Z_{ci}) - \hat{H}^*(Z_{ci})] \big(\hat{\phi}_i - \phi_i \big) [H^*(Z_{mi}) - \hat{H}^*(Z_{mi})] \Big)^2 \\ &\le p \max_{1 \le a \le p} \sup_{X,V \in G_{XV}} |H^*(Z_a) - \hat{H}^*(Z_a)|^4 \sup_{X,V \in G_{XV}} |\hat{\phi}(X,V) - \phi(X,V)| \\ &= O_p(\mathcal{L}_n^4) O_p(\mathcal{L}_{0n}^2) = o_p(1). \end{aligned}$$

Consequently, $||A_8|| = \text{trace}(A_8 A_8')^{1/2} = \Big[\sum_{c=1}^p A_8(c,:)' A_8(c,:) \Big]^{1/2} = o_p(1)$. In all,

$$||A - \Sigma_0|| \le ||A_1 - \Sigma_0|| + ||A_2|| + ||A_3|| + ||A_4|| + ||A_5|| + ||A_6|| + ||A_7|| + ||A_8|| = o_p(1).$$

Consequently $A = \Sigma_0 + o_p(1)$. Now, recall from Lemma 1 that $\beta_0 = \mu_Y - \mu_Z' \beta_1$ and consider,

$$\begin{aligned} Y_i - \hat{H}^*(Y_i) - \big(Z_i - \hat{H}^*(Z_i) \big)' \beta_1 &= Y_i - Z_i \beta_1 - \big(\hat{H}^*(Y_i) - \hat{H}^*(Z_i)' \beta_1 \big) \\ &= Y_i - Z_i' \beta_1 - \beta_0 - h(X_i) - f(V_i) + \beta_0 + h(X_i) + f(V_i) - \big(\hat{H}^*(Y_i) - \hat{H}^*(Z_i)' \beta_1 \big) \\ &= u_i + \beta_0 + \sum_{d=1}^D \big[H_1^d(Y_i) - \beta_0 - H_1^d(Z_i)' \beta_1 \big] + \sum_{d=1}^D \big[H_2^d(Y_i) - \beta_0 - H_2^d(Z_i)' \beta_1 \big] \\ &\quad - \Big\{ \sum_{d=1}^D \big[\hat{H}_1^d(Y_i) - \hat{H}_1^d(Z_i)' \beta_1 \big] + \sum_{d=1}^D \big[\hat{H}_2^d(Y_i) - \hat{H}_2^d(Z_i)' \beta_1 \big] + (2D-1) \big[\hat{\mu}_Z' \beta_1 - \hat{\mu}_Y \big] \Big\} \\ &= u_i + \sum_{d=1}^D \big[H_1^d(Y_i) - \hat{H}_1^d(Y_i) \big] + \sum_{d=1}^D \big[H_2^d(Y_i) - \hat{H}_2^d(Y_i) \big] - \sum_{d=1}^D \big[H_1^d(Z_i) - \hat{H}_1^d(Z_i) \big]' \beta_1 \\ &\quad - \sum_{d=1}^D \big[H_2^d(Z_i) - \hat{H}_2^d(Z_i) \big]' \beta_1 - (2D-1) \big[\mu_Y - \hat{\mu}_Y \big] + (2D-1) \big[\mu_Z - \hat{\mu}_Z \big]' \beta_1 \\ &= u_i + \sum_{d=1}^D \big[\mathsf{S}_1^d(Y_i) + \mathsf{S}_2^d(Y_i) \big] - \sum_{d=1}^D \big[\mathsf{S}_1^d(Z_i) + \mathsf{S}_2^d(Z_i) \big] \beta_1 + (2D-1) \Big(\big[\mu_Z - \hat{\mu}_Z \big]' \beta_1 - \big[\mu_Y - \hat{\mu}_Y \big] \Big). \end{aligned}$$

In vector notation,

$$\begin{aligned} \mathbf{Y}_n - \hat{\mathbf{H}}_n^*(Y) - \big(\hat{\mathbf{Z}}_n - \hat{\mathbf{H}}_n^*(Z) \big) \beta_1 = \mathbf{u}_n &+ \sum_{d=1}^D \big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z) \beta_1 \big] + \sum_{d=1}^D \big[\mathsf{S}_{2n}^d(Y) - \mathsf{S}_{2n}^d(Z) \beta_1 \big] \\ &+ (2D-1) \Big(\big[\boldsymbol{\mu}_{Zn} - \hat{\boldsymbol{\mu}}_{Z_n} \big]' \beta_1 - \big[\boldsymbol{\mu}_{Yn} - \hat{\boldsymbol{\mu}}_{Yn} \big] \Big). \end{aligned}$$

Furthermore,

$$\begin{aligned} B &= n^{-1} \big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z) \big]' \hat{\boldsymbol{\phi}}_n \big[\mathbf{Y}_n - \hat{\mathbf{H}}_n^*(Y) - \big(\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z) \big) \beta_1 \big] \\ &= n^{-1} \big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z) \big]' \hat{\boldsymbol{\phi}}_n \mathbf{u}_n \end{aligned}$$

$$
\begin{aligned}
&+\sum_{d=1}^{D} n^{-1}\big[\mathbf{Z}_n-\hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathsf{S}_{1n}^d(Y)-\mathsf{S}_{1n}^d(Z)\beta_1\big]\\
&+\sum_{d=1}^{D} n^{-1}\big[\mathbf{Z}_n-\hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathsf{S}_{2n}^d(Y)-\mathsf{S}_{2n}^d(Z)\beta_1\big]\\
&+(2D-1)n^{-1}\big[\mathbf{Z}_n-\hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\Big(\big[\boldsymbol{\mu}_{Zn}-\hat{\boldsymbol{\mu}}_{Zn}\big]'\beta_1-\big[\boldsymbol{\mu}_{Yn}-\hat{\boldsymbol{\mu}}_{Yn}\big]\Big)\\
\equiv{}& B_1+\sum_{d=1}^{D}B_{2d}+\sum_{d=1}^{D}B_{3d}+B_4.
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= n^{-1}\big[\mathbf{Z}_n-\hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\mathbf{u}_n\\
&= n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\mathbf{u}_n+n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\mathbf{u}_n\\
&\quad+n^{-1}\big[\mathbf{H}_n^*(Z)-\hat{\mathbf{H}}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\mathbf{u}_n+n^{-1}\big[\mathbf{H}_n^*(Z)-\hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\mathbf{u}_n\\
&\equiv B_{11}+B_{12}+B_{13}+B_{14}.
\end{aligned}
$$

Consider, $B_{11}=n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\mathbf{u}_n=n^{-1}\sum_{i=1}^n[Z_i-H^*(Z_i)]\phi_iu_i$. Now, since, $\{Z_i,X_i,V_i\}_{i=1}^n$ is i.i.d,

$$
E\big[\big(Z_i-H^*(Z_i)\big)\phi_iu_i\big]=E\big[\big(Z_i-H^*(Z_i)\big)\phi_iE\big(u_i|Z_i,X_i,V_i\big)\big]=0,
$$

and, $V\big[\big(Z_i-H^*(Z_i)\big)\phi_iu_i\big]=E\big[E\big(u_i^2|Z_i,X_i,V_i\big)\big(Z_i-H^*(Z_i)\big)\phi_i\big(Z_i-H^*(Z_i)\big)'\big]=\Sigma_1=O(1)$. Consequently by CLT,

$$
\sqrt{n}B_{11}=n^{-1/2}\sum_{i=1}^n[Z_i-H^*(Z_i)]\phi_iu_i\xrightarrow{d}N\big(0,\Sigma_1\big).
$$

By Lemma 10,

$$
B_{12}=n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\mathbf{u}_n=n^{-1}\sum_{i=1}^n\big[Z_i-H^*(Z_i)\big]u_i\big(\hat{\phi}_i-\phi_i\big)=n^{-1}\sum_{i=1}^n\zeta_iu_i\big(\hat{\phi}_i-\phi_i\big)=o_p(n^{-1/2}).
$$

By Theorem's 1,2 and Lemma 3 xxvi) and xxvii),

$$
\begin{aligned}
B_{13}&=n^{-1}\big[\mathbf{H}_n^*(Z)-\hat{\mathbf{H}}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\mathbf{u}_n=n^{-1}\sum_{i=1}\big[H^*(Z_i)-\hat{H}^*(Z_i)\big]\big(\hat{\phi}_i-\phi_i\big)u_i.\\
&\le \sup_{XV\in G_{XV}}|H^*(Z)-\hat{H}^*(Z)|\sup_{XV\in G_{XV}}|\hat{\phi}(X,\hat{V})-\phi(X,V)|n^{-1}\sum_{i=1}^n|u_i|\\
&=O_p(\mathcal{L}_n)O_p(\mathcal{L}_{0n})O_p(1)=o_p(n^{-1/2}).
\end{aligned}
$$

$$
\begin{aligned}
B_{14}&=n^{-1}\big[\mathbf{H}_n^*(Z)-\hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\mathbf{u}_n\\
&=n^{-1}\sum_{i=1}^n\phi_iu_i\Big[\sum_{d=1}^D(-1)\big[\hat{H}_1^d(Z_i)-H_1^d(Z_i)\big]-\sum_{d=1}^D\big[\hat{H}_2^d(Z_i)-H_2^d(Z_i)\big]+(2D+1)\big[\big(\mu_Z-\hat{\mu}_Z\big)'\beta_1-(\mu_Y-\hat{\mu}_Y)\big]\Big]\\
&=(-1)\sum_{d=1}^Dn^{-1}\sum_{i=1}^n\phi_iu_i\big[\hat{H}_1^d(Z_i)-H_1^d(Z_i)\big]-\sum_{d=1}^Dn^{-1}\sum_{i=1}^n\phi_iu_i\big[\hat{H}_2^d(Z_i)-H_2^d(Z_i)\big]\\
&\qquad+(2D+1)\big[\big(\mu_Z-\hat{\mu}_Z\big)'\beta_1-(\mu_Y-\hat{\mu}_Y)\big]n^{-1}\sum_{i=1}^n\phi_iu_i\\
&=(-1)\sum_{d=1}^DB_{141d}-\sum_{d=1}^DB_{142d}+B_{143}.
\end{aligned}
$$

For $c \in \{1, 2, \cdots, p\}$ consider,

$$
\begin{aligned}
B_{141d}(c) &= n^{-1} \sum_{i=1}^{n} \phi_i u_i \big[\hat{H}_1^d(Z_{ci}) - H_1^d(Z_{ci})\big] \\
&= n^{-1} \sum_{i=1}^{n} \phi_i u_i \left[[(n-1)b_1]^{-1} \sum_{j \neq i} K_{1ji}(X_d) \hat{\theta}_{1j}^d Z_{cj} - H_1^d(Z_{ci}) \right] \\
&= n^{-1} \sum_{i=1}^{n} \phi_i u_i [(n-1)b_1]^{-1} \sum_{j \neq i} K_{1ji}(X_d) \big(\hat{\theta}_{1j}^d - \theta_{1j}^d\big) Z_{cj} \\
&\quad + n^{-1} \sum_{i=1}^{n} \phi_i u_i [(n-1)b_1]^{-1} \sum_{j \neq i} K_{1ji}(X_d) \theta_{1j}^d \big(Z_{cj} - E[Z_{cj}|X_j, V_j]\big) \\
&\quad + n^{-1} \sum_{i=1}^{n} \phi_i u_i [(n-1)b_1]^{-1} \sum_{j \neq i} K_{1ji}(X_d) \theta_{1j}^d \big(E[Z_{cj}|X_j, V_j] - H_1^d(Z_{cj})\big) \\
&\quad + n^{-1} \sum_{i=1}^{n} \phi_i u_i [(n-1)b_1]^{-1} \sum_{j \neq i} K_{1ji}(X_d) \theta_{1j}^d \big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big) \\
&\equiv B_{1411d}(c) + B_{1412d}(c) + B_{1413d}(c) + B_{1414d}(c).
\end{aligned}
$$

By Lemma 3, Assumption A5, and Theorem 1,

$$
\begin{aligned}
B_{1411d} &= n^{-1} \sum_{i=1}^{n} \phi_i u_i [(n-1)b_1]^{-1} \sum_{j \neq i}^{n} K_{1ji}(X_d) \big(\hat{\theta}_{1j}^d - \theta_{1j}^d\big) Z_{cj} \\
&= (n-1)^{-1} \sum_{j \neq i} \big(\hat{\theta}_{1j}^d - \theta_{1j}^d\big) Z_{cj} [nb_1]^{-1} \sum_{i=1}^{n} \phi_i u_i K_{1ji}(X_d) \\
&\leq \sup_{X, V \in G_{XV}} |\hat{\theta}_1^d(X, \hat{V}) - \theta_1^d(X, V)| (n-1)^{-1} \sum_{j \neq i} |Z_{cj}| \\
&\qquad \times \sup_{X_{dj} \in G_{X_d}} \left| [(n-1)b_1]^{-1} \sum_{i=1}^{n} \big(\phi_i u_i K_{1ji}(X_d) - E\big[\phi_i u_i K_{1ji}(X_d)\big]\big) \right| \\
&= O_p(\mathcal{L}_{0n}) O_p\left(\left[\frac{\log(n)}{nb_1} \right]^{1/2} \right) O(1) = o_p(n^{-1/2}).
\end{aligned}
$$

Note that $E\big[\rho_{cj}|X_j, V_j\big] = E\big[Z_{cj} - E\big(Z_{cj}|X_j, V_j\big)\big|X_j, V_j\big] = 0$ and consider,

$$
\begin{aligned}
B_{1412d}(c) &= n^{-1} \sum_{i=1}^{n} \phi_i u_i [(n-1)b_1]^{-1} \sum_{j \neq i} K_{1ji}(X_d) \theta_{1j}^d \big(Z_{cj} - E[Z_{cj}|X_j, V_j]\big) \\
&= [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \phi_i u_i b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \rho_{cj} \\
&= [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \Psi_{1412}^d(i, j; c) \simeq \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{i<j} \Gamma_{1412}^{(2)d}(i, j; c) = U_{1412}^{(2)d}(c).
\end{aligned}
$$

where $\Gamma_{1412}^{(2)d}(i, j; c) = \Psi_{1412}^d(i, j; c) + \Psi_{1412}^d(j, i; c)$.

$$E(\Psi_{1412}^d(i, j; c)) = E\big(b_1^{-1} \phi_i u_i K_{1ji}(X_d) \theta_{1j}^d \rho_{cj}\big) = E\big(b_1^{-1} \phi_i K_{1ji}(X_d) \theta_{1j}^d \rho_{cj} E[u_i|Z_i, X_i, V_i, S_j]\big) = 0.$$

$$E[E(\Psi^d_{1412}(i,j;c)|S_i)^2]^{1/2} = E\big[E\big(b_1^{-1}\phi_i u_i K_{1ji}(X_d)\theta^d_{1j} E\big[\rho_{cj}|X_j,V_j,S_i\big]|S_i\big)^2\big]^{1/2} = 0.$$

$$E[E(\Psi^d_{1412}(i,j;c)|S_j)^2]^{1/2} = E\big[E\big(b_1^{-1}\phi_i K_{1ji}(X_d)\theta^d_{1j}\rho_{cj} E\big[u_i|Z_i,X_i,V_i,S_j\big]\big|S_j\big)^2\big]^{1/2} = 0.$$

$$\begin{aligned}
E(\Psi^d_{1412}(i,j;c)^2)^{1/2} &= E\big(b_1^{-2}\phi_i^2 u_i^2 K_{1ji}(X_d)^2[\theta^d_{1j}]^2\rho^2_{cj}\big)^{1/2} \\
&\leq \sup_{X,V\in G_{XV}} \big|\phi(X,V)\theta^d_1(X,V)\big| E\big(b_1^{-2}E\big[u_i^2|Z_i,X_i,V_i,S_j\big]K_{1ji}(X_d)^2 E\big[\rho^2_{cj}|X_{dj},S_i\big]\big)^{1/2} \\
&= O(b_1^{-1/2})E\big(b_1^{-1}K_{1ji}(X_d)^2\big)^{1/2} = O(b_1^{-1/2}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
B_{1412d}(c) &= E\big(\Psi^d_{1412}(i,j;c)\big) + O_p\big(n^{-1/2}E[E(\Psi^d_{1412}(i,j;c)|S_i)^2]^{1/2}\big) \\
&\quad + O_p\big(n^{-1/2}E[E(\Psi^d_{1412}(i,j;c)|S_j)^2]^{1/2}\big) + O_p\big(n^{-1}E\big(\Psi^d_{1412}(i,j;c)^2\big)^{1/2}\big) \\
&= O_p\big([nb_1^{1/2}]^{-1}\big) = o_p(n^{-1/2}).
\end{aligned}$$

by assumption A5.

$$\begin{aligned}
B_{1413d}(c) &= n^{-1}\sum_{i=1}^n \phi_i u_i[(n-1)b_1]^{-1}\sum_{j\neq i} K_{1ji}(X_d)\theta^d_{1j}\big(E[Z_{cj}|X_j,V_j] - H^d_1(Z_{cj})\big) \\
&= [n(n-1)]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i K_{1ji}(X_d)\theta^d_{1j}\eta^d_{1cj} \\
&= [n(n-1)]^{-1}\sum_{i=1}^n\sum_{j\neq i}\Psi^d_{1413}(i,j;c) \simeq \binom{n}{2}^{-1}\sum_{i=1}^n\sum_{i<j}\Gamma^{(2)d}_{1413}(i,j;c) = U^{(2)d}_{1413}(c).
\end{aligned}$$

where $\Gamma^{(2)d}_{1413}(i,j;c) = \Psi^d_{1413}(i,j;c) + \Psi^d_{1413}(j,i;c)$

$$E(\Psi^d_{1413}(i,j;c)) = E\big(b_1^{-1}\phi_i u_i K_{1ji}(X_d)\theta^d_{1j}\eta^d_{1cj}\big) = E\big(b_1^{-1}\phi_i K_{1ji}(X_d)\theta^d_{1j}\eta^d_{1cj}E[u_i|Z_i,X_i,V_i,S_j]\big) = 0.$$

$$E[E(\Psi^d_{1413}(i,j;c)|S_i)^2]^{1/2} = E\big(\phi_i^2u_i^2 E\big[b_1^{-1}\theta^d_{1j}K_{1ji}(X_d)\eta^d_{1cj}\big|S_j\big]^2\big)^{1/2} = O(b_1^{\nu_1})E\big(\phi_i^2u_i^2\big)^{1/2} = O(b_1^{\nu_1}),$$

by Lemma 7.

$$E[E(\Psi^d_{1413}(i,j;c)|S_j)^2]^{1/2} = E\big[E\big(b_1^{-1}\phi_i K_{1ji}(X_d)\theta^d_{1j}\eta^d_{1cj}E\big[u_i|Z_i,X_i,V_i,S_j\big]\big|S_j\big)^2\big]^{1/2} = 0.$$

$$\begin{aligned}
2E(\Psi^d_{1413}(i,j;c)^2)^{1/2} &= 2E\big(b_1^{-2}\phi_i^2u_i^2K_{1ji}(X_d)^2[\theta^d_{1j}]^2[\eta^d_{1cj}]^2\big)^{1/2} \\
&\leq \sup_{XV\in G_{XV}}|\phi(X,V)\theta^d_1(X,V)\eta^d_{1c}(X,V)|b_1^{-1/2}E\big[b_1^{-1}K_{1ji}(X_d)^2E\big(u_i|Z_i,X_i,V_i,S_j\big)\big]^{1/2} \\
&= O(b_1^{-1/2})E\big[b_1^{-1}K_{1ji}(X_d)^2\big]^{1/2} = O(b_1^{-1/2}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
B_{1413d}(c) &\simeq U^{(2)d}_{1413}(c) \\
&= E(\Psi^d_{1413}(i,j;c)) + O_p\big(n^{-1/2}E[E(\Psi^d_{1413}(i,j;c)|S_i)^2]^{1/2}\big) \\
&\quad + O_p\big(n^{-1/2}E[E(\Psi^d_{1413}(i,j;c)|S_j)^2]^{1/2}\big) + O_p\big(n^{-1}E(\Psi^d_{1413}(i,j;c)^2)^{1/2}\big)
\end{aligned}$$

$$= O_p(n^{-1/2}b_1^{\nu_1}) + O_p(n^{-1}b_1^{-1/2}) = o_p(n^{-1/2}).$$

$$\begin{aligned}
B_{1414d}(c) &= n^{-1}\sum_{i=1}^{n}\phi_i u_i[(n-1)b_1]^{-1}\sum_{j\neq i}K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big) \\
&= [n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}b_1^{-1}\phi_i u_i K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big) \\
&= [n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_{1414}^d(i,j;c) \simeq \binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_{1414}^{(2)d}(i,j;c) = U_{1414}^{(2)d}(c).
\end{aligned}$$

where $\Gamma_{1414}^{(2)d}(i,j;c) = \Psi_{1414}^d(i,j;c) + \Psi_{1414}^d(j,i;c)$

$$\begin{aligned}
E\big(\Psi_{1414}^d(i,j;c)\big) &= E\big(b_1^{-1}\phi_i u_i K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)\big) \\
&= E\big(b_1^{-1}\phi_i K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)E[u_i|Z_i, X_i, V_i, S_j]\big) = 0.
\end{aligned}$$

$$\begin{aligned}
E[E\big(\Psi_{1414}^d(i,j;c)|S_i\big)^2]^{1/2} &= E\big[E\big(b_1^{-1}\phi_i u_i K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)|S_i\big)^2\big]^{1/2} \\
&= E\big[\phi_i^2 u_i^2 E\big(b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)|S_i\big)^2\big]^{1/2} \\
&\leq O(b_1^{\nu_1})E\big[\phi_i^2 u_i^2\big]^{1/2} = O(b_1^{\nu_1}),
\end{aligned}$$

by Lemma 7.

$$\begin{aligned}
E[E\big(\Psi_{1414}^d(i,j;c)|S_i\big)^2]^{1/2} &= E\big[E\big(b_1^{-1}\phi_i u_i K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)|S_j\big)^2\big]^{1/2} \\
&= \big[E\big(b_1^{-1}\phi_i K_{1ji}(X_d)\theta_{1j}^d\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)E[u_i|Z_i, X_i, V_i, S_j]|S_j\big)^2\big]^{1/2} = 0.
\end{aligned}$$

$$\begin{aligned}
E\big(\Psi_{1414}^d(i,j;c)^2\big)^{1/2} &= E\big(b_1^{-2}\phi_i^2 u_i^2 K_{1ji}(X_d)^2[\theta_{1j}^d]^2\big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)^2\big)^{1/2} \\
&\leq \sup_{X,V\in G_{XV}}|\phi(X,V)\theta_1^d(X,V)| \sup_{X_d,X_d'\in G_{X_d}}|H_1^d(Z_c;X_d) - H_1^d(Z_c;X_d')| \\
&\qquad \times E\big[b_1^{-2}K_{1ji}(X_d)^2E\big(u_i^2|Z_i, X_i, V_i, S_j\big)\big]^{1/2} \\
&= O(b_1^{-1/2})E\big[b_1^{-1}K_{1ji}(X_d)^2\big]^{1/2} = O(b_1^{-1/2}).
\end{aligned}$$

Consequently, by Lemma 3,

$$\begin{aligned}
B_{1414d}(c) &\simeq U_{1414}^{(2)d}(c) \\
&= E\big(\Psi_{1414}^d(i,j;c)\big) + O_p\big(n^{-1/2}E[E\big(\Psi_{1414}^d(i,j;c)|S_i\big)^2]^{1/2}\big) \\
&\quad + O_p\big(n^{-1/2}E[E\big(\Psi_{1414}^d(i,j;c)|S_j\big)^2]^{1/2}\big) + O_p\big(n^{-1}E\big(\Psi_{1414}^d(i,j;c)^2\big)^{1/2}\big) \\
&= O_p(n^{-1/2}b_1^{\nu_1}) + O_p(n^{-1}b_1^{-1/2}) = o_p(n^{-1/2}).
\end{aligned}$$

Furthermore, $B_{141d}(c) = B_{1411d}(c) + B_{1412d}(c) + B_{1413d}(c) + B_{1414d}(c) = o_p(n^{-1/2})$.

$$\begin{aligned}
B_{142d}(c) &= n^{-1}\sum_{i=1}^{n}\phi_i u_i\big[\hat{H}_2^d(Z_{ci}) - H_2^d(Z_{ci})\big] \\
&= n^{-1}\sum_{i=1}^{n}\phi_i u_i\Big\{[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(\hat{V}_d)\hat{\theta}_{2j}^d Z_{cj} - H_2^d(Z_{ci})\Big\} \\
&= n^{-1}\sum_{i=1}^{n}\phi_i u_i\Big\{[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(\hat{V}_d)\big(\hat{\theta}_{2j}^d - \theta_{2j}^d + \theta_{2j}^d\big)Z_{cj} - H_2^d(Z_{ci})\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= n^{-1}\sum_{i=1}^{n}\phi_i u_i[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(\hat{V}_d)\Big\{\big(\hat{\theta}_{2j}^d-\theta_{2j}^d\big)Z_{cj}+\theta_{2j}^d\big(Z_{cj}-E[Z_{cj}|X_j,V_j]\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\theta_{2j}^d\big(E[Z_{cj}|X_j,V_j]-H_2^d(Z_{cj})\big)+\theta_{2j}^d\big(H_2^d(Z_{cj})-H_2^d(Z_{ci})\big)\Big\}\\
&= n^{-1}\sum_{i=1}^{n}\phi_i u_i[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}[b_2^{-1}(\hat{V}_{dj}-\hat{V}_{di})]\mathsf{C}_{2ji}^d(c)\\
&= n^{-1}\sum_{i=1}^{n}\phi_i u_i[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(V_d)\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i u_i[(n-1)b_2^2]^{-1}\sum_{j\neq i}K_{2ji}^{(1)}(V_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})]\,\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i u_i[2(n-1)b_2^3]^{-1}\sum_{j\neq i}K_{2ji}^{(2)}(V_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})]^2\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i u_i[6(n-1)b_2^4]^{-1}\sum_{j\neq i}K_{2ji}^{(3)}(V_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})]^3\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i u_i[24(n-1)b_2^5]^{-1}\sum_{j\neq i}K_{2ji}^{(4)}(\tilde{V}_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})]^4\mathsf{C}_{2ji}^d(c)\\
&= [n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}(V_d)\mathsf{C}_{2ji}^d(c)\\
&\quad-[n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_j)-m_d(W_j)]\,\mathsf{C}_{2ji}^d(c)\\
&\quad+[n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)]\,\mathsf{C}_{2ji}^d(c)\\
&\quad+[2n(n-1)b_2^2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(2)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)]^2\mathsf{C}_{2ji}^d(c)\\
&\quad+[6n(n-1)b_2^3]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(3)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)]^3\mathsf{C}_{2ji}^d(c)\\
&\quad+[24n(n-1)b_2^5]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i K_{2ji}^{(4)}(\tilde{V}_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)]^4\mathsf{C}_{2ji}^d(c)\\
&\equiv B_{1421d}(c)-B_{1422d}(c)+B_{1423d}(c)+B_{1424d}(c)+B_{1425d}(c)+B_{1426d}(c).
\end{aligned}
$$

The proof of the order of $B_{1421d}(c)$ is, mutatis mutandis, virtually identical to that of $B_{1411d}(c)$ and as such, the arguments will not be repeated here. Thus, one can conclude, $B_{1421d}(c) = o_p(n^{-1/2})$. Note that,

$$
E\Big[\phi_i u_i H_2^d(Z_{ci})b_2^{-1}K_{2ji}^{(1)}(V_d)\Big] = E\Big[\phi_i u_i H_2^d(Z_{ci})b_2^{-1}K_{2ji}^{(1)}(V_d)E(u_i|Z_i,V_i,X_i,S_j)\Big] = 0,
$$

and also, $\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d) = K_{2ji}^{(1)}(V_d)\theta_{2i}^d\big[p(V_{di})u_i H_2^d(Z_{ci})\big]$ where,

$$
E\big[|p(V_{di})u_i H_2^d(Z_{ci})|^2|V_{di}\big] \le \Big(\sup_{V_d\in G_{V_d}} p(V_d)|H_2^d(Z_{ci})|\Big)^2 E\big[u_i^2|Z_i,X_i,V_i,S_j\big] < \infty,
$$

Consequently, by a trivial modification of Lemma 6,

$$
\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^{n}\Big(\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d)-E\Big[\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d)\Big]\Big)\Big| = O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right).
$$

In the trivial case where one sets $H_2^d(Z_{ci}) = 1$,

$$\sup_{V_{dj}\in G_{V_d}} \Big|[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i K_{2ji}^{(1)}(V_d)\Big]\Big)\Big| = O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right).$$

$$\begin{aligned}
B_{1422d}(c) &= [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat m_d^{l_n}(W_j) - m_d(W_j)]\,\mathsf{C}_{2ji}^d(c) \\
&= (n-1)^{-1}\sum_{j\neq i}[\hat m_d^{l_n}(W_j) - m_d(W_j)]\big(\hat\theta_{2j}^d - \theta_{2j}^d\big)Z_{cj}[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i K_{2ji}^{(1)}(V_d)\Big]\Big) \\
&\quad + (n-1)^{-1}\sum_{j\neq i}[\hat m_d^{l_n}(W_j) - m_d(W_j)]\theta_{2j}^d\rho_{cj}[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i K_{2ji}^{(1)}(V_d)\Big]\Big) \\
&\quad + (n-1)^{-1}\sum_{j\neq i}[\hat m_d^{l_n}(W_j) - m_d(W_j)]\theta_{2j}^d\eta_{2cj}^d[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i K_{2ji}^{(1)}(V_d)\Big]\Big) \\
&\quad + (n-1)^{-1}\sum_{j\neq i}[\hat m_d^{l_n}(W_j) - m_d(W_j)]\theta_{2j}^d H_2^d(Z_{cj})[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i K_{2ji}^{(1)}(V_d)\Big]\Big) \\
&\quad - (n-1)^{-1}\sum_{j\neq i}[\hat m_d^{l_n}(W_j) - m_d(W_j)]\theta_{2j}^d[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d)\Big]\Big) \\
&\le \sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i K_{2ji}^{(1)}(V_d)\Big]\Big)\Big| \sup_{W\in G_W}\big|\hat m_d^{l_n}(W) - m_d(W)\big| \\
&\qquad\times\Big\{\sup_{XV\in G_{XV}}|\hat\theta_2^d(X,\hat V) - \theta_2^d(X,V)|(n-1)^{-1}\sum_{j\neq i}|Z_{cj}| \\
&\qquad\qquad + \sup_{XV\in G_{XV}}\theta_2^d(X,V)(n-1)^{-1}\sum_{j\neq i}\big(|\rho_{cj}| + |\eta_{2cj}^d| + |H_2^d(Z_{cj})|\big)\Big\} \\
&\quad + \sup_{XV\in G_{XV}}\theta_2^d(X,V)\sup_{W\in G_W}\big|\hat m_d^{l_n}(W) - m_d(W)\big| \\
&\qquad\times\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^n \Big(\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d) - E\Big[\phi_i u_i H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d)\Big]\Big)\Big| \\
&= O_p(L_n)O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right)\Big\{O_p(\mathcal{L}_{0n}) + O(1)\Big\} = O_p(L_n b_2^{-1})O_p\left(\left[\frac{\log(n)}{nb_2^{1/2}}\right]^{1/2}\right) = o_p(n^{-1/2}),
\end{aligned}$$

by Lemmas 3 and 6. Consider

$$\begin{aligned}
B_{1423d}(c) &= [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat m_d^{l_n}(W_i) - m_d(W_i)]\,\mathsf{C}_{2ji}^d(c) \\
&= [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat m_d^{l_n}(W_i) - m_d(W_i)](\hat\theta_{2j}^d - \theta_{2j}^d)Z_{cj} \\
&\quad + [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat m_d^{l_n}(W_i) - m_d(W_i)]\theta_{2j}^d\rho_{cj} \\
&\quad + [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat m_d^{l_n}(W_i) - m_d(W_i)]\theta_{2j}^d\eta_{2cj}^d \\
&\quad + [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i u_i b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat m_d^{l_n}(W_i) - m_d(W_i)]\theta_{2j}^d\big(H_2^d(Z_{cj}) - H_2^d(Z_{ci})\big)
\end{aligned}$$

$$= (n-1)^{-1} \sum_{j \neq i} \mathsf{C}_{2j}^{d*}(c)[nb_2^2]^{-1} \sum_{i=1}^{n} \phi_i u_i K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)]$$

$$- (n-1)^{-1} \sum_{j \neq i} \theta_{2j}^{d}[nb_2^2]^{-1} \sum_{i=1}^{n} \phi_i u_i H_2^d(Z_{ci}) K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)]$$

$$= (n-1)^{-1} \sum_{j=1}^{n} \mathsf{C}_{2j}^{d*}(c)[nb_2^2]^{-1} \sum_{i \neq j} \phi_i u_i K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)]$$

$$- (n-1)^{-1} \sum_{j=1}^{n} \theta_{2j}^{d}[nb_2^2]^{-1} \sum_{i \neq j} \phi_i u_i H_2^d(Z_{ci}) K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)]$$

$$= (n-1)^{-1} \sum_{j=1}^{n} \mathsf{C}_{2j}^{d*}(c)[nb_2^2]^{-1} \mathbf{K}_{2j}(V_d)' \boldsymbol{\phi}_n \dot{\boldsymbol{u}}_n I(-j)[\hat{\mathbf{M}}_d^{l_n} - \mathbf{M}_d]$$

$$- (n-1)^{-1} \sum_{j=1}^{n} \theta_{2j}^{d}[nb_2^2]^{-1} \mathbf{K}_{2j}(V_d)' \boldsymbol{\phi}_n \dot{\boldsymbol{u}}_n \dot{\mathbf{H}}_2^d(Z_c) I(-j)\big[\hat{\mathbf{M}}_d^{l_n} - \mathbf{M}_d\big]$$

$$\leq n^{-1} \sum_{j=1}^{n} \left|\mathsf{C}_{2j}^{d*}(c)\right| \, ||[(n-1)b_2^2]^{-1} \mathbf{K}_{2j}(V_d)' \boldsymbol{\phi}_n \dot{\boldsymbol{u}}_n I(-j) \mathbf{B}_n||_E O_p\left(\sqrt{\frac{l_n}{n}} + l_n^{-k}\right)$$

$$- [n(n-1)]^{-1} \sum_{j=1}^{n} \sum_{i \neq j} \mathsf{C}_{2j}^{d*}(c) \phi_i u_i b_2^{-2} K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)]$$

$$- n^{-1} \sum_{j=1}^{n} ||[(n-1)b_2^2]^{-1} \mathbf{K}_{2j}(V_d)' \boldsymbol{\phi}_n \dot{\boldsymbol{u}}_n \dot{\mathbf{H}}_2^d(Z_c) I(-j) \mathbf{B}_n||_E O_p\left(\sqrt{\frac{l_n}{n}} + l_n^{-k}\right)$$

$$+ [n(n-1)]^{-1} \sum_{j=1}^{n} \sum_{i \neq j} \phi_i u_i b_2^{-2} H_2^d(Z_{ci}) K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)]$$

$$\equiv B_{14231d}(c) + B_{14232d}(c) + B_{14233d}(c) + B_{14234d}(c).$$

Note that,

$$B_{14231d}(c) = n^{-1} \sum_{j=1}^{n} \left|\mathsf{C}_{2j}^{d*}(c)\right| \, ||[(n-1)b_2^2]^{-1} \mathbf{K}_{2j}(V_d)' \boldsymbol{\phi}_n \dot{\boldsymbol{u}}_n I(-j) \mathbf{B}_n||_E O_p\left(\sqrt{\frac{l_n}{n}} + l_n^{-k}\right).$$

consider,

$$E\big(||[(n-1)b_2^2]^{-1} \mathbf{K}_{2j}(V_d)' \boldsymbol{\phi}_n \dot{\boldsymbol{u}}_n I(-j) \mathbf{B}_n||_E^2\big) = E\Big(||[(n-1)b_2^2]^{-1} \sum_{i \neq j} \mathbf{B}_n(W_i) K_{2ji}(V_d) \phi_i u_i||_E\Big)$$

$$= E\Big([(n-1)b_2^2]^{-2} \sum_{i \neq j} \sum_{g \neq j} \mathbf{B}_n(W_g)' \mathbf{B}_n(W_i) K_{2gi}(V_d) K_{2ji}(V_d) \phi_g u_g \phi_i u_i\Big)$$

$$= [(n-1)^2 b_2^3]^{-1} \sum_{i \neq j} E\Big(\mathbf{B}_n(W_i)' \mathbf{B}_n(W_i) b_2^{-1} K_{2ji}(V_d)^2 \phi_j^2 E\big[u_j^2 | W_j, X_j, V_j, S_i\big]\Big)$$

$$+ [(n-1)b_2^2]^{-2} \sum_{i \neq j} \sum_{\substack{g \neq j \\ g \neq i}} E\Big(\mathbf{B}_n(W_g)' \mathbf{B}_n(W_i) K_{2gi}(V_d) K_{2ji}(V_d) \phi_g \phi_i E[u_g | W_g, X_g, V_g, S_{-g}] E[u_i | W_i, X_i, V_i, S_{-i}]\Big)$$

$$\leq \sup_{X, V \in G_{XV}} |\phi(X, V)| O([(n-1)b_2^3]^{-1}) E\Big(\mathbf{B}_n(W_i)' \mathbf{B}_n(W_i) E[b_2^{-1} K_{2ji}(V_d)^2 | S_i]\Big)$$

$$= O([(n-1)b_2^3]^{-1}) E\Big(\mathbf{B}_n(W_i)' \mathbf{B}_n(W_i)\Big) = O\left(\frac{l_n}{(n-1)b_2^3}\right).$$

Consequently, by Lemma 3, and Markov's Inequality,

$$
\begin{aligned}
B_{14231d}(c) &= O_p\left(\frac{l_n}{nb_2^{3/2}}\right) n^{-1}\sum_{j=1}^n \left|\mathsf{C}_{2j}^{d*}(c)\right| \\
&= o_p(n^{-1/2})n^{-1}\sum_{j=1}^n \left|\mathsf{C}_{2j}^{d*}(c)\right| \\
&\leq o_p(n^{-1/2})\Big\{\sup_{XV\in G_{XV}} |\hat{\theta}_2^d(X,\hat{V}) - \theta_2^d(X,V)|n^{-1}\sum_{j=1}^n |Z_{cj}| \\
&\qquad + n^{-1}\sum_{j=1}^n |\theta_{2j}^d\rho_{cj}| + n^{-1}\sum_{j=1}^n |\theta_{2j}^d\eta_{2cj}^d| + n^{-1}\sum_{j=1}^n |\theta_{2j}^d H_2^d(Z_{cj})|\Big\} \\
&= o_p(n^{-1/2}).
\end{aligned}
$$

Let $Q_{2j}^d(c) = \rho_{cj} + \eta_{2cj}^d + H_2^d(Z_{cj})$ and note that,

$$
\begin{aligned}
B_{14232d}(c) &= [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}\mathsf{C}_{2j}^{d*}(c)\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&= [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}(\hat{\theta}_{2j}^d - \theta_{2j}^d)Z_{cj}\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&\quad + [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}\theta_{2j}^d\rho_{cj}\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&\quad + [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}\theta_{2j}^d\eta_{2cj}^d\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&\quad + [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}\theta_{2j}^d H_2^d(Z_{cj})\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&\leq \sup_{X,V\in G_{XV}} |\hat{\theta}_2^d(X,\hat{V}) - \theta_2^d(X,V)| \sup_{W\in G_W} |m_d^{l_n}(W) - m_d(W)| \\
&\qquad \times n^{-1}\sum_{j=1}^n |Z_{cj}|[(n-1)b_2]^{-1}\sum_{i\neq j}|\phi_i u_i|\left|b_2^{-1}K_{2ji}^{(1)}(V_d)\right| \\
&\quad + [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}\theta_{2j}^d\left[\rho_{cj} + \eta_{2cj}^d + H_2^d(Z_{cj})\right]\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&\equiv O_p(\mathcal{L}_{0n})O(L_n)Op(b_2^{-1}) + [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}^n\theta_{2j}^d Q_{2j}^d(c)\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)] \\
&\equiv o_p(n^{-1/2}) + B_{142321d}(c).
\end{aligned}
$$

In the following, it is important to note that by A4 and A5,

$$
\sup_{V_{dj}\in G_{V_d}} E\big(\theta_{2j}^d Q_{2j}^d(c)|V_d\big) \leq \sup_{X,V\in G_{XV}} \theta_2^d(X,V)\Big[\sup_{V_{dj}\in G_{V_d}} E\big(|\rho_{cj}||V_{dj}\big) + \sup_{V_{dj}\in G_{V_d}} E\big(|\eta_{2cj}^d||V_{dj}\big) + \sup_{V_{dj}\in G_{V_d}} |H_2^d(Z_{cj})|\Big] = O(1)
$$

and

$$
\sup_{V_{dj}\in G_{V_{dj}}} E\big([\theta_{2j}^d Q_{2j}^d(c)]^2|V_{dj}\big) = \sup_{X,V\in G_{XV}} \theta_2^d(X,V)^2 E\Big[\big(\rho_{cj} + \eta_{2cj}^d + H_2^d(Z_{cj})\big)^2\Big|V_{dj}\Big] = O(1).
$$

$$
B_{142321d}(c) = [n(n-1)]^{-1}\sum_{j=1}^n\sum_{i\neq j}\theta_{2j}^d Q_{2j}^d(c)\phi_i u_i b_2^{-2}K_{2ji}^{(1)}(V_d)[m_d^{l_n}(W_i) - m_d(W_i)]
$$

$$= [n(n-1)]^{-1}\sum_{j=1}^{n}\sum_{i\neq j}\Psi^d_{142321}(j,i;c) \simeq \binom{n}{2}^{-1}\sum_{j=1}^{n}\sum_{i<j}\Gamma^{(2)d}_{142321}(j,i;c) = U^{(2)d}_{142321}(c).$$

where $\Gamma^{(2)d}_{142321}(j,i;c) = \Psi^d_{142321}(j,i;c) + \Psi^d_{142321}(i,j;c)$

$$\begin{aligned} E\Big(\Psi^d_{142321}(j,i;c)\Big) &= E\Big(\theta^d_{2j}Q^d_{2j}(c)\phi_i u_i b_2^{-2}K^{(1)}_{2ji}(V_d)[m^{l_n}_d(W_i) - m_d(W_i)]\Big) \\ &= E\Big(\theta^d_{2j}Q^d_{2j}(c)\phi_i b_2^{-2}K^{(1)}_{2ji}(V_d)[m^{l_n}_d(W_i) - m_d(W_i)]E\big[u_i|W_i,X_i,V_i,S_j\big]\Big) = 0. \end{aligned}$$

$$\begin{aligned} E\big[E\big(\Psi^d_{142321}(j,i;c)|S_i\big)^2\big]^{1/2} &= E\Big[\Big(\theta^d_{2j}Q^d_{2j}(c)\phi_i u_i b_2^{-2}K^{(1)}_{2ji}(V_d)[m^{l_n}_d(W_i) - m_d(W_i)]\big|S_i\Big)^2\Big]^{1/2} \\ &\leq \sup_{W\in G_W}|m^{l_n}_d(W) - m_d(W)|E\Big[\phi_i^2u_i^2E\Big(b_2^{-2}K^{(1)}_{2ji}(V_d)E\big(\theta^d_{2j}Q^d_{2j}(c)|V_{dj},S_i\big)\big|S_i\Big)^2\Big]^{1/2} \\ &\leq O(L_nb_2^{-1})E\Big[\phi_i^2u_i^2E\Big(b_2^{-1}|K^{(1)}_{2ji}(V_d)|\Big|S_i\Big)^2\Big]^{1/2} \\ &(L_nb_2^{-1})E[\phi_i^2u_i^2]^{1/2} = O(L_nb_2^{-1}). \end{aligned}$$

$$\begin{aligned} E\big[E\big(\Psi^d_{142321}(j,i;c)|S_j\big)^2\big]^{1/2} &= E\Big[\Big(\theta^d_{2j}Q^d_{2j}(c)\phi_i u_i b_2^{-2}K^{(1)}_{2ji}(V_d)[m^{l_n}_d(W_i) - m_d(W_i)]\big|S_j\Big)^2\Big]^{1/2} \\ &= E\Big[\Big(\theta^d_{2j}Q^d_{2j}(c)\phi_i b_2^{-2}K^{(1)}_{2ji}(V_d)[m^{l_n}_d(W_i) - m_d(W_i)]E\big(u_i|W_i,X_i,V_i,S_j\big)\big|S_j\Big)^2\Big]^{1/2} = 0. \end{aligned}$$

$$\begin{aligned} E\Big(\Psi^d_{142321}(j,i;c)^2\Big)^{1/2} &= E\Big([\theta^d_{2j}Q^d_{2j}(c)]^2\phi_i^2u_i^2b_2^{-4}K^{(1)}_{2ji}(V_d)^2[m^{l_n}_d(W_i) - m_d(W_i)]^2\Big)^{1/2} \\ &\leq \sup_{W\in G_W}|m^{l_n}_d(W) - m_d(W)|E\Big(E\big([\theta^d_{2j}Q^d_{2j}(c)]^2\big|V_{dj}\big)\phi_i^2u_i^2b_2^{-4}K^{(1)}_{2ji}(V_d)^2\Big)^{1/2} \\ &= O(L_nb_2^{-3/2})E\Big(\phi_i^2u_i^2E\Big[b_2^{-1}K^{(1)}_{2ji}(V_d)^2\Big|S_i\Big]\Big)^{1/2} \\ &= O(L_nb_2^{-3/2})E\Big(\phi_i^2u_i^2\Big)^{1/2} = O(L_nb_2^{-3/2}). \end{aligned}$$

Consequently,

$$\begin{aligned} B_{142321d}(c) = U^{(2)d}_{142321}(c) &= E\big(\Psi^d_{142321}(j,i;c)\big) + O_p\Big(n^{-1/2}E\big[E\big(\Psi^d_{142321}(j,i;c)|S_i\big)^2\big]^{1/2}\Big) \\ &+ O_p\Big(n^{-1/2}E\big[E\big(\Psi^d_{142321}(j,i;c)|S_j\big)^2\big]^{1/2}\Big) + O_p\Big(n^{-1}E\big(\Psi^d_{142321}(j,i;c)^2\big)^{1/2}\Big) \\ &= O_p(n^{-1/2}L_nb_2^{-1/2}) + O_p(n^{-1}L_nb_2^{-3/2}) = o_p(n^{-1/2}), \end{aligned}$$

by Lemma 3, Consequently,

$$B_{14232d}(c) = o_p(n^{-1/2}) + B_{142321d}(c) = o_p(n^{-1/2}).$$

The proof of the order of $B_{14231d}(c)$, and $B_{14232d}(c)$ immediately gives,

$$B_{14233d}(c) + B_{14234d}(c) = o_p(n^{-1/2}).$$

As a result,

$$B_{1423d}(c) = B_{14231d}(c) + B_{14232d}(c) + B_{14233d}(c) + B_{14234d}(c) = o_p(n^{-1/2}).$$

By Lemma 3 vi) and vii) we note that,

$$
\begin{aligned}
&B_{1424d}(c)+B_{1425d}(c)+B_{1426d}(c)\\
&\quad=[2n(n-1)b_2^2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i K_{2ji}^{(2)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^2\mathsf{C}_{2ji}^d(c)\\
&\qquad+[6n(n-1)b_2^3]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i K_{2ji}^{(3)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^3\mathsf{C}_{2ji}^d(c)\\
&\qquad+[24n(n-1)b_2^5]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i u_i K_{2ji}^{(4)}(\tilde{V}_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^4\mathsf{C}_{2ji}^d(c)\\
&\quad\le\sup_{W\in G_W}2^2|\hat{m}_d^{l_n}(W)-m_d(W)|^2[2n(n-1)b_2^2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\qquad+\sup_{W\in G_W}2^3|\hat{m}_d^{l_n}(W)-m_d(W)|^3[6n(n-1)b_2^3]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\qquad+\sup_{W\in G_W}2^4|\hat{m}_d^{l_n}(W)-m_d(W)|^4\sup_{\gamma\in\mathbb{R}}\big|K_{2ji}^{(4)}(\gamma)\big|[24n(n-1)b_2^5]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\quad=O_p\Big([L_n b_2^{-1}]^2\Big)[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\qquad+O_p\Big([L_n b_2^{-1}]^3\Big)[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\qquad+O_p\Big([L_n^4 b_2^{-5}]\Big)[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\quad=o_p(n^{-1/2})[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\Big(\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|+\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|+1\Big)\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\quad\le o_p(n^{-1/2})[n(n-1)]^{-1}\sup_{X,V\in G_{XV}}|\hat{\theta}_2^d(X,\hat{V})-\theta_2^d(X,V)|\\
&\qquad\times\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\Big(\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|+\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|+1\Big)|Z_{cj}|\\
&\qquad+o_p(n^{-1/2})[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\Big(\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|+\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|+1\Big)\big|\theta_{2j}^d Q_{2j}^d(c)\big|\\
&\qquad+o_p(n^{-1/2})[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_i u_i|\Big(\big|b_2^{-1}K_{2ji}^{(2)}(V_d)\big|+\big|b_2^{-1}K_{2ji}^{(3)}(V_d)\big|+1\Big)\big|\theta_{2j}^d H_2^d(Z_{ci})\big|\\
&\quad=o_p(n^{-1/2})O_p(\mathcal{L}_{0n})[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_{1424}^d(i,j;c)\\
&\qquad+o_p(n^{-1/2})[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_{1425}^d(i,j;c)\\
&\qquad+o_p(n^{-1/2})[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_{1426}^d(i,j;c)\\
&\quad\simeq o_p(n^{-1/2})\Big\{O_p(\mathcal{L}_{0n})\binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_{1424}^d(i,j;c)+\binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_{1425}^d(i,j;c)+\binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_{1426}^d(i,j;c)\Big\}\\
&\quad=o_p(n^{-1/2})\Big\{O_p(\mathcal{L}_{0n})U_{1424}^{(2)d}(c)+U_{1425}^{(2)d}(c)+U_{1426}^{(2)d}(c)\Big\}.
\end{aligned}
$$

where, $\Gamma^d_{1424}(i,j;c) = \Psi^d_{1424}(i,j;c) + \Psi^d_{1424}(j,i;c)$, $\Gamma^d_{1425}(i,j;c) = \Psi^d_{1425}(j,i;c) + \Psi^d_{1425}(i,j;c)$, and $\Gamma^d_{1426}(i,j;c) = \Psi^d_{1426}(i,j;c) + \Psi^d_{1426}(j,i;c)$, Now note,

$$\begin{aligned}
&E\big(\Psi^d_{1424}(i,j;c)\big) + E\big(\Psi^d_{1425}(i,j;c)\big) + E\big(\Psi^d_{1426}(i,j;c)\big)\\
&\quad = E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]|Z_{cj}|\Big)\\
&\qquad + E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}Q^d_{2j}(c)\big|\Big)\\
&\qquad + E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}H^d_2(Z_{ci})\big|\Big)\\
&\quad \le E\Big(|\phi_i u_i||Z_{cj}|E\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1|Z_{cj}, S_i\Big]\Big)\\
&\qquad + E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]E\Big[\big|\theta^d_{2j}Q^d_{2j}(c)\big|\Big|V_{dj}, S_i\Big]\Big)\\
&\qquad + \sup_{X,V\in G_{X,V}}\big|\theta^d_2(X,V)E[\phi Z_c|V_d]\big|E\Big(|\phi_i u_i|E\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big|S_i\Big]\Big)\\
&\quad = O(1)E\big(|\phi_i u_i|\big)E\big(|Z_{cj}|\big) + O(1)E\Big(|\phi_i u_i|E\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big|S_i\Big]\Big) + O(1)E\big(|\phi_i u_i|\big)\\
&\quad = O(1) + O(1)E\big(|\phi_i u_i|\big) + O(1) = O(1).
\end{aligned}$$

$$\begin{aligned}
&E\big[E\big(\Psi^d_{1424}(i,j;c)|S_i\big)^2\big]^{1/2} + E\big[E\big(\Psi^d_{1425}(i,j;c)|S_i\big)^2\big]^{1/2} + E\big[E\big(\Psi^d_{1426}(i,j;c)|S_i\big)^2\big]^{1/2}\\
&\quad = E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]|Z_{cj}|\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\qquad + E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}Q^d_{2j}(c)\big|\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\qquad + E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}H^d_2(Z_{ci})\big|\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\quad = E\Big[|\phi_i u_i|^2E\Big(|Z_{cj}|E\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big|Z_{cj}, S_i\Big]\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\qquad + E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]E\Big[\big|\theta^d_{2j}Q^d_{2j}(c)\big|\Big|V_{dj}, S_i\Big]\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\qquad + \sup_{X,V\in G_{X,V}}\big|\theta^d_2(X,V)E[\phi Z_c|V_d]E\Big[|\phi_i u_i|^2E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\quad = O(1)E\Big[|\phi_i u_i|^2E\Big(|Z_{cj}|\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\qquad + O(1)E\Big[|\phi_i u_i|^2E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\Big|\, S_i\Big)^2\Big]^{1/2}\\
&\qquad + O(1)E\Big[|\phi_i u_i|^2\Big]^{1/2}\\
&\quad = O(1)E\Big[|\phi_i u_i|^2\Big]^{1/2} + O(1)E\Big[|\phi_i u_i|^2\Big]^{1/2} + O(1)\\
&\quad = O(1) + O(1) + O(1).
\end{aligned}$$

$$\begin{aligned}
&E\big[E\big(\Psi^d_{1424}(i,j;c)|S_j\big)^2\big]^{1/2} + E\big[E\big(\Psi^d_{1425}(i,j;c)|S_j\big)^2\big]^{1/2} + E\big[E\big(\Psi^d_{1426}(i,j;c)|S_j\big)^2\big]^{1/2}\\
&\quad = E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]|Z_{cj}|\Big|\, S_j\Big)^2\Big]^{1/2}\\
&\qquad + E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}Q^d_{2j}(c)\big|\Big|\, S_j\Big)^2\Big]^{1/2}\\
&\qquad + E\Big[E\Big(|\phi_i u_i|\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}H^d_2(Z_{ci})\big|\Big|\, S_j\Big)^2\Big]^{1/2}\\
&\quad \le \sup_{X,V\in G_{XV}}\phi(X,V)\Big\{E\Big[|Z_{cj}|^2E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]E\big(|u_i|\big|Z_i, X_i, V_i, S_j\big)\Big|\, S_j\Big)^2\Big]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+ E\Big[\big|\theta^d_{2j}Q^d_{2j}(c)\big|^2 E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]E\big(|u_i|\big|Z_i,X_i,V_i,S_j\big)\Big|\ S_j\Big)^2\Big]^{1/2} \\
&+ \sup_{X,V\in G_{X,V}} \big|\theta^d_2(X,V)E\big[\phi Z_c|V_d\big]\big|E\Big[E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]E\big(|u_i|\big|Z_i,X_i,V_i,S_j\big)\Big|\ S_j\Big)^2\Big]^{1/2}\Big\} \\
\le{}& O(1)E\Big[|Z_{cj}|^2 E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\Big|\ S_j\Big)^2\Big]^{1/2} \\
&+ O(1)E\Big[\big|\theta^d_{2j}Q^d_{2j}(c)\big|^2 E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\Big|\ S_j\Big)^2\Big]^{1/2} \\
&+ O(1)E\Big[E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\Big|\ S_j\Big)^2\Big]^{1/2} \\
={}& O(1)E\big[|Z_{cj}|^2\big] + O(1)\big[\big|\theta^d_{2j}Q^d_{2j}(c)\big|^2\big] + O(1) \\
={}& O(1) + O(1) + O(1).
\end{aligned}
$$

Where

$$
\begin{aligned}
&E\big[\Psi^d_{1424}(i,j;c)^2\big]^{1/2} + E\big[\Psi^d_{1425}(i,j;c)^2\big]^{1/2} + E\big[\Psi^d_{1426}(i,j;c)^2\big]^{1/2} \\
&= E\Big[\phi_i^2u_i^2\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]^2 Z^2_{cj}\Big]^{1/2} \\
&\quad + E\Big[\phi_iu_i|^2\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]^2\big|\theta^d_{2j}Q^d_{2j}(c)\big|^2\Big]^{1/2} \\
&\quad + E\Big[|\phi_iu_i|^2\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]\big|\theta^d_{2j}H^d_2(Z_{ci})\big|\Big]^{1/2} \\
&= \sup_{X,V\in G_{XV}} |\phi(X,V)|\Big\{E\Big[E(u_i^2|Z_i,X_i,V_i,S_j)E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]^2\Big|Z_{cj},S_i\Big)Z^2_{cj}\Big]^{1/2} \\
&\quad + E\Big[E(u_i^2|Z_i,X_i,V_i,S_j)\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]^2E\Big(\big|\theta^d_{2j}Q^d_{2j}(c)\big|^2\Big|V_{dj},S_i\Big)\Big]^{1/2} \\
&\quad + \sup_{X,V\in G_{X,V}} \big|\theta^d_2(X,V)E\big[\phi Z_c|V_d\big]\big|E\Big[E(u_i^2|Z_i,X_i,V_i,S_j)E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]^2\Big]^{1/2}\Big\} \\
&= O(b_2^{1/2})E[Z^2_{cj}]^{1/2} + O(1)E\Big(\Big[\big|b_2^{-1}K^{(2)}_{2ji}(V_d)\big| + \big|b_2^{-1}K^{(3)}_{2ji}(V_d)\big| + 1\Big]^2\Big)^{1/2} + O(b_2^{1/2}) \\
&= O(b_2^{-1/2}) + O(b_2^{-1/2}) + O(b_2^{-1/2}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
U^{(2)d}_{1424} + U^{(2)d}_{1425}(c) + U^{(2)d}_{1426}(c) ={}& E\big(\Psi^d_{1424}(c)(i,j;c)\big) + E\big(\Psi^d_{1425}(i,j;c)\big) + E\big(\Psi^d_{1426}(i,j;c)\big) \\
&+ O_p\big(n^{-1/2}E\big[E\big(\Psi^d_{1424}(i,j;c)|S_i\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi^d_{1425}(i,j;c)|S_i\big)^2\big]^{1/2}\big) \\
&+ O_p\big(n^{-1/2}E\big[E\big(\Psi^d_{1426}(i,j;c)|S_i\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi^d_{1424}(i,j;c)|S_j\big)^2\big]^{1/2}\big) \\
&+ O_p\big(n^{-1/2}E\big[E\big(\Psi^d_{1425}(i,j;c)|S_j\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi^d_{1426}(i,j;c)|S_j\big)^2\big]^{1/2}\big) \\
&+ O_p\big(n^{-1}E\big[\Psi^d_{1424}(i,j;c)^2\big]^{1/2}\big) + O_p\big(n^{-1}E\big[\Psi^d_{1425}(i,j;c)^2\big]^{1/2}\big) \\
&+ O_p\big(n^{-1}E\big[\Psi^d_{1426}(i,j;c)^2\big]^{1/2}\big) \\
={}& O(1) + O_p(n^{1/2}) + O_p(n^{-1}b_2^{-1/2}) = O(1).
\end{aligned}
$$

In all,

$$
\begin{aligned}
B_{1424d}(c) + B_{1425d}(c) + B_{1426d}(c) &= o_p(n^{-1/2})\Big\{O_p(\mathcal{L}_{0n})U^{(2)d}_{1424}(c) + U^{(2)d}_{1425}(c) + U^{(2)d}_{1426}(c)\Big\} \\
&= o_p(n^{-1/2})O_p(\mathcal{L}_{0n} + 2) = o_p(n^{-1/2}).
\end{aligned}
$$

Furthermore,

$$
B_{142d}(c) = B_{1421d}(c) + B_{1422d}(c) + B_{1423d}(c) + B_{1424d}(c) + B_{1425d}(c) + B_{1426d}(c) = o_p(n^{1/2}).
$$

Now,

$$
\begin{aligned}
B_{143} &= (2D+1)\big[(\mu_Z - \hat{\mu}_Z)\beta_1 - (\mu_Y - \hat{\mu}_Y)\big] n^{-1}\sum_{i=1}^{n}\phi_i u_i \\
&\le (2D+1)\Big[|\mu_Y - \hat{\mu}_Y| + \max_{1\le c\le p}|\mu_{Z_c} - \hat{\mu}_{Z_c}|\sum_{c=1}^{p}|\beta_1|\Big]\Big|n^{-1}\sum_{i=1}^{n}\phi_i u_i\Big| \\
&= O_p\big(\mathcal{L}_{0n}\big)\big(1 + ||\beta_1||_E\big)\Big|\sum_{i=1}^{n}\phi_i u_i\Big| = O_p\big(\mathcal{L}_{0n}\big)O(1)o_p(n^{-1/2}) = o_p(n^{-1/2}).
\end{aligned}
$$

Hence,

$$
B_{14} = (-1)\sum_{d=1}^{D} B_{141d} - \sum_{d=1}^{D} B_{142d} + B_{143} = o_p(n^{-1/2}).
$$

Thus in summary,

$$
\sqrt{n}B_1 = \sqrt{n}B_{11} + \sqrt{n}B_{12} + \sqrt{n}B_{13} + \sqrt{n}B_{14} = \sqrt{n}B_{11} + o_p(1).
$$

Consequently, $\sqrt{n}B_1 \xrightarrow{d} N(0, \Sigma_1)$.

$$
\begin{aligned}
B_2 &= \sum_{d=1}^{D} n^{-1}\big[\mathbf{Z}_n - \hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] \\
&= \sum_{d=1}^{D} n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z) + \mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\big(\boldsymbol{\phi}_n + \hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] \\
&= \sum_{d=1}^{D}\Big\{ n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] + n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] \\
&\qquad + n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] + n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big]\Big\} \\
&\equiv \sum_{d=1}^{D}\big[B_{21d} + B_{22d} + B_{23d} + B_{24d}\big].
\end{aligned}
$$

$$
\begin{aligned}
B_{22d} &= n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] \\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i - H^*(Z_i)\big]\big(\hat{\phi}_i - \phi_i\big)\Big(H_1^d(Y_i) - \hat{H}_1^d(Y_i) - \sum_{c=1}^{p}\big[H_1^d(Z_{ci}) - \hat{H}_1^d(Z_{ci})\big]\beta_{1c}\Big) \\
&\le \sup_{XV\in G_{XV}}|\hat{\phi}(X,\hat{V}) - \phi(X,V)|\Big[\sup_{X_d\in G_{X_d}}\big|H_1^d(Y_i) - \hat{H}_1^d(Y_i)\big| \\
&\qquad\qquad + \max_{1\le j\le p}\sup_{X_d\in G_{X_d}}\big|H_1^d(Z_j) - \hat{H}_1^d(Z_j)\big|\sum_{c=1}^{p}|\beta_{1c}|\Big] n^{-1}\sum_{i=1}^{n}\big|Z_i - H^*(Z_i)\big| \\
&\le O_p\big(\mathcal{L}_{0n}\big)O_p\big(\mathcal{L}_{1n}\big)\big(1 + ||\beta_1||_E\big)O_p(1) = O_p\big(\mathcal{L}_{0n}\mathcal{L}_{1n}\big) = o_p(n^{-1/2}).
\end{aligned}
$$

By Lemma 3 xxvi) and xxvii), Assumption A3, and Markov's inequality.

$$
\begin{aligned}
B_{23d}(c) &= n^{-1}\big[\mathbf{H}_n^*(Z_c) - \hat{\mathbf{H}}_n^*(Z_c)\big]'\boldsymbol{\phi}_n\big[\mathsf{S}_{1n}^d(Y) - \mathsf{S}_{1n}^d(Z)\beta_1\big] \\
&= n^{-1}\sum_{i=1}^{n}\big[H^*(Z_{ci}) - \hat{H}^*(Z_{ci})\big]\phi_i\Big(H_1^d(Y_i) - \hat{H}_1^d(Y_i) - \sum_{c=1}^{p}\big[H_1^d(Z_{ci}) - \hat{H}_1^d(Z_{ci})\big]\beta_{1c}\Big) \\
&\le \sup_{XV\in G_{XV}}|H^*(Z_c) - \hat{H}^*(Z_c)|\Big[\sup_{X_d\in G_{X_d}}\big|H_1^d(Y) - \hat{H}_1^d(Y)\big|
\end{aligned}
$$

$$+\max_{1\leq j\leq p}\sup_{X_d\in G_{X_d}}\left|H_1^d(Z_j)-\hat{H}_1^d(Z_j)\right|\sum_{c=1}^{p}|\beta_{1c}|\Big]n^{-1}\sum_{i=1}^{n}\phi_i$$

$$\leq O_p\big(\mathcal{L}_n\mathcal{L}_{1n}\big)\big(1+||\beta_1||_E\big)O(1)=O_p\big(\mathcal{L}_n\mathcal{L}_{1n}\big)=o_p(n^{-1/2}),$$

by Lemma 3 xxvi) and xxvii), Assumption A3, and Markov's Inequality.

$$\begin{aligned}
B_{24d}(c) &= n^{-1}\big[\mathbf{H}_n^*(Z_c)-\hat{\mathbf{H}}_n^*(Z_c)\big]'\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\big[\mathsf{S}_{1n}^d(Y)-\mathsf{S}_{1n}^d(Z)\beta_1\big]\\
&= n^{-1}\sum_{i=1}^{n}\big[H^*(Z_{ci})-\hat{H}^*(Z_{ci})\big]\big(\hat{\phi}_i-\phi_i\big)\Big(H_1^d(Y_i)-\hat{H}_1^d(Y_i)-\sum_{c=1}^{p}\big[H_1^d(Z_{ci})-\hat{H}_1^d(Z_{ci})\big]\beta_{1c}\Big)\\
&\leq \sup_{XV\in G_{XV}}|\hat{\phi}(X,\hat{V})-\phi(X,V)|\sup_{XV\in G_{XV}}|H^*(Z_c)-\hat{H}^*(Z_c)|\Big[\sup_{X_d\in G_{X_d}}\left|H_1^d(Y)-\hat{H}_1^d(Y)\right|\\
&\qquad +\max_{1\leq j\leq p}\sup_{X_d\in G_{X_d}}\left|H_1^d(Z_j)-\hat{H}_1^d(Z_j)\right|\sum_{c=1}^{p}|\beta_{1c}|\Big]\\
&= O_p\big(\mathcal{L}_n\big)O_p\big(\mathcal{L}_{0n}\big)O_p\big(\mathcal{L}_{1n}\big)\big(1+||\beta_1||_E\big)=O_p\big(\mathcal{L}_n\big)O_p\big(\mathcal{L}_{0n}\big)O_p\big(\mathcal{L}_{1n}\big)=o_p(n^{-1/2}),
\end{aligned}$$

by Lemma 3 xxvi) and xxvii), and Assumption A3.

$$\begin{aligned}
B_{21d} &= n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]\boldsymbol{\phi}_n\big[\mathsf{S}_{1n}^d(Y)-\mathsf{S}_{1n}^d(Z)\beta_1\big]\\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\phi_i\Big(H_1^d(Y_i)-\hat{H}_1^d(Y_i)-\sum_{c=1}^{p}\big[H_1^d(Z_{ci})-\hat{H}_1^d(Z_{ci})\big]\beta_{1c}\Big)\\
&= (-1)n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\phi_i\big(\hat{H}_1^d(Y_i)-H_1^d(Y_i)\big)\\
&\qquad +\sum_{c=1}^{p}\beta_{1c}\,n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\phi_i\big[\hat{H}_1^d(Z_{ci})-H_1^d(Z_{ci})\big]\\
&= B_{211d}-\sum_{c=1}^{p}\beta_{1c}B_{212dc}.
\end{aligned}$$

Note that by Lemma 2, $E[\phi_i\zeta_{ai}K_{1ji}(X_d)|X_{di},S_j]=K_{1ji}(X_d)E[\phi_i\zeta_{ai}|X_{di}]=0$. Furthermore, since by Assumption 4, $E\big[\phi_i^2\zeta_{ci}^2|X_{di}\big]=p(X_d)^2E\big[[\theta_{1i}^d]^2\zeta_{ci}^2|X_{di}\big]$ is uniformly bounded so that we have by Lemma 6,

$$\sup_{X_{di}\in G_{X_d}}\left|n^{-1}\sum_{i=1}^{n}\Big[\phi_i\zeta_{ai}b_1^{-1}K_{1ji}(X_d)-E\Big(\phi_i\zeta_{ai}b_1^{-1}K_{1ji}(X_d)\Big)\Big]\right|=O_p\left(\left[\frac{\log(n)}{nb_1}\right]^{1/2}\right).$$

Consider,

$$\begin{aligned}
B_{212dc}(a) &= n^{-1}\sum_{i=1}^{n}\big[Z_{ai}-H^*(Z_{ai})\big]\phi_i\big[\hat{H}_1^d(Z_{ci})-H_1^d(Z_{ci})\big]\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}\Big\{[(n-1)b_1]^{-1}\sum_{j\neq i}K_{1ji}(X_d)\hat{\theta}_{1j}^dZ_{ci}-H_1^d(Z_{ci})\Big\}\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[(n-1)b_1]^{-1}\sum_{j\neq i}K_{1ji}(X_d)\Big[\big(\hat{\theta}_{1j}^d-\theta_{1j}^d\big)Z_{cj}+\theta_{1j}^d\big(Z_{cj}-E[Z_{cj}|X_j,V_j]\big)\\
&\qquad +\theta_{1j}^d\big(E[Z_{cj}|X_j,V_j]-H_1^d(Z_{cj})\big)+\theta_{1j}^d\big(H_1^d(Z_{cj})-H_1^d(Z_{ci})\big)\Big]\\
&= [n(n-1)]^{-1}\sum_{j\neq i}\big(\hat{\theta}_{1j}^d-\theta_{1j}^d\big)Z_{cj}\sum_{i=1}^{n}\phi_i\zeta_{ai}b_1^{-1}K_{1ji}(X_d)\\
&\qquad +[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_1^{-1}K_{1ji}(X_d)\theta_{1j}^d\rho_{cj}
\end{aligned}$$

$$
\begin{aligned}
&+ [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \phi_i \zeta_{ai} b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \eta_{1cj}^d \\
&+ [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \phi_i \zeta_{ai} b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big) \\
\equiv{}& [n-1]^{-1} \sum_{j \neq i} \big(\hat{\theta}_{1j}^d - \theta_{1j}^d\big) Z_{cj} n^{-1} \sum_{i=1}^{n} \Big[\phi_i \zeta_{ai} b_1^{-1} K_{1ji}(X_d) - E\Big(\phi_i \zeta_{ai} b_1^{-1} K_{1ji}(X_d)\Big)\Big] \\
&+ [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \Psi_{2121}^{dc}(i,j;a) + [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \Psi_{2122}^{dc}(i,j;a) \\
&+ [n(n-1)]^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \Psi_{2123}^{dc}(i,j;a) \\
\lesssim{}& \sup_{XV \in G_{XV}} |\hat{\theta}_1^d(X,\hat{V}) - \theta_1^d(X,V)| \sup_{X_{di} \in G_{X_d}} \Big|[nb_1]^{-1} \sum_{j \neq i} \Big[\phi_i \zeta_{ai} K_{1ji}(X_d) - E\Big(\phi_i \zeta_{ai} K_{1ji}(X_d)\Big)\Big| [n-1]^{-1} \sum_{j \neq i} |Z_{cj}| \\
&+ \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{i<j} \Gamma_{2121}^{dc}(i,j;a) + \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{i<j} \Gamma_{2122}^{dc}(i,j;a) + \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{i<j} \Gamma_{2123}^{dc}(i,j;a) \\
={}& O_p\big(\mathcal{L}_{0n}\big) O_p\left(\left[\frac{\log(n)}{nb_1}\right]^{1/2}\right) O_p(1) + U_{2121}^{(2)dc}(a) + U_{2122}^{(2)dc}(a) + U_{2123}^{(2)dc}(a) \\
={}& o_p(n^{-1/2}) + U_{2121}^{(2)dc}(a) + U_{2122}^{(2)dc}(a) + U_{2123}^{(2)dc}(a),
\end{aligned}
$$

by Lemma 3 xxi), and xxvi) where, $\Gamma_{2121}^{dc}(i,j;a) = \Psi_{2121}^{dc}(i,j;a) + \Psi_{2121}^{dc}(j,i;a)$, $\Gamma_{2122}^{dc}(i,j;a) = \Psi_{2122}^{dc}(i,j;a) + \Psi_{2122}^{dc}(i,j;a)$, and $\Gamma_{2123}^{dc}(i,j;a) = \Psi_{2123}^{dc}(i,j;a) + \Psi_{2123}^{dc}(j,i;a)$

$$
\begin{aligned}
E\big[\Psi_{2121}^{dc}(i,j;a)\big] &+ E\big[\Psi_{2122}^{dc}(i,j;a)\big] + E\big[\Psi_{2123}^{dc}(i,j;a)\big] \\
&= E\big[E\big(\phi_i \zeta_{ai} | X_{di}, S_j\big) b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \rho_{cj}\big] \\
&\quad + E\big[E\big(\phi_i \zeta_{ai} | X_{di}, S_j\big) b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \eta_{1cj}^d\big] \\
&\quad + E\big[E\big(\phi_i \zeta_{ai} | X_{di}, S_j\big) b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)\big] \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
E\big[E\big(\Psi_{2121}^{dc}(i,j;a)|S_i\big)^2\big]^{1/2} &+ E\big[E\big(\Psi_{2122}^{dc}(i,j;a)|S_i\big)^2\big]^{1/2} + E\big[E\big(\Psi_{2123}^{dc}(i,j;a)|S_i\big)^2\big]^{1/2} \\
&= E\big[E\big(\phi_i \zeta_{ai} b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d E\big[\rho_{cj}|X_j, V_j, S_i\big]|S_i\big)^2\big]^{1/2} \\
&\quad + E\big[\phi_i^2 \zeta_{ai}^2 E\big(b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \eta_{1cj}^d |S_i\big)^2\big]^{1/2} \\
&\quad + E\big[\phi_i^2 \zeta_{ai}^2 E\big(b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)|S_i\big)^2\big]^{1/2} \\
&= 0 + O(b_1^{\nu_1}) E\big[\phi_i^2 \zeta_{ai}^2\big]^{1/2} + O(b_1^{\nu_1}) E\big[\phi_i^2 \zeta_{ai}^2\big]^{1/2} \\
&= 0 + O(b_1^{\nu_1}) + O(b_1^{\nu_1}),
\end{aligned}
$$

by Lemma 8.

$$
\begin{aligned}
E\big[E\big(\Psi_{2121}^{dc}(i,j;a)|S_j\big)^2\big]^{1/2} &+ E\big[E\big(\Psi_{2122}^{dc}(i,j;a)|S_j\big)^2\big]^{1/2} + E\big[E\big(\Psi_{2123}^{dc}(i,j;a)|S_j\big)^2\big]^{1/2} \\
&= E\big[E\big(E\big[\phi_i \zeta_{ai}|X_{di}, S_j\big] b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \rho_{cj}\big|S_j\big)^2\big]^{1/2} \\
&\quad + E\big[E\big(E\big[\phi_i \zeta_{ai}|X_{di}, S_j\big] b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \eta_{1cj}^d |S_j\big)^2\big]^{1/2} \\
&\quad + E\big[E\big(E\big[\phi_i \zeta_{ai}|X_{di}, S_j\big] b_1^{-1} K_{1ji}(X_d) \theta_{1j}^d \big(H_1^d(Z_{cj}) - H_1^d(Z_{ci})\big)|S_j\big)^2\big]^{1/2} \\
&= 0,
\end{aligned}
$$

by Lemma 2.

$$
\begin{aligned}
E\big[\Psi^{dc}_{2121}(i,j;a)^2\big]^{1/2} &+ E\big[\Psi^{dc}_{2122}(i,j;a)^2\big]^{1/2} + E\big[\Psi^{dc}_{2123}(i,j;a)^2\big]^{1/2} \\
&= E\big[E\big(\phi_i^2\zeta_{ai}^2|X_{di},S_j\big)b_1^{-2}K_{1ji}(X_d)^2[\theta^d_{1j}\rho_{cj}]^2\big]^{1/2} \\
&\quad + E\big[E\big(\phi_i^2\zeta_{ai}^2|X_{di},S_j\big)b_1^{-2}K_{1ji}(X_d)^2[\theta^d_{1j}\eta^d_{1cj}]^2\big] \\
&\quad + E\big[E\big(\phi_i^2\zeta_{ai}^2|X_{di},S_j\big)b_1^{-2}K_{1ji}(X_d)^2[\theta^d_{1j}]^2\big(H^d_1(Z_{cj}) - H^d_1(Z_{ci})^2\big)\big]^{1/2} \\
&\le O\big(b_1^{-1/2}\big)E\big[E\big(b_1^{-1}K_{1ji}(X_d)^2|S_j\big)[\theta^d_{1j}\rho_{cj}]^2\big]^{1/2} \\
&\quad + \big(b_1^{-1/2}\big)E\big[E\big(b_1^{-1}K_{1ji}(X_d)^2|S_j\big)[\theta^d_{1j}\eta^d_{1cj}]^2\big]^{1/2} \\
&\quad + \big(b_1^{-1/2}\big)\sup_{X_d\in G_{X_d}} 2E\big[\phi Z_c\big|X_d\big]^2 E\big[E\big(b_1^{-1}K_{1ji}(X_d)^2|S_j\big)[\theta^d_{1j}]^2\big]^{1/2} \\
&= O\big(b_1^{-1/2}\big)E\big[(\theta^d_{1j}\rho_{cj})^2\big]^{1/2} + O\big(b_1^{-1/2}\big)E\big[(\theta^d_{1j}\eta^d_{1cj})^2\big]^{1/2} + O\big(b_1^{-1/2}\big) \\
&= O\big(b_1^{-1/2}\big) + O\big(b_1^{-1/2}\big) + O\big(b_1^{-1/2}\big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
U^{(2)dc}_{2121}(a) &+ U^{(2)dc}_{2122}(a) + U^{(2)dc}_{2123}(a) \\
&= E\big[\Psi^{dc}_{2121}(i,j;a)\big] + E\big[\Psi^{dc}_{2122}(i,j;a)\big] + E\big[\Psi^{dc}_{2123}(i,j;a)\big] \\
&\quad + O_p\big(n^{-1/2}E\big[E\big(\Psi^{dc}_{2121}(i,j;a)|S_i\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi^{dc}_{2122}(i,j;a)|S_i\big)^2\big]^{1/2}\big) \\
&\quad + O_p\big(n^{-1/2}E\big[E\big(\Psi^{dc}_{2123}(i,j;a)|S_i\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi^{dc}_{2121}(i,j;a)|S_j\big)^2\big]^{1/2}\big) \\
&\quad + O_p\big(n^{-1/2}E\big[E\big(\Psi^{dc}_{2122}(i,j;a)|S_j\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi^{dc}_{2123}(i,j;a)|S_j\big)^2\big]^{1/2}\big) \\
&\quad + O_p\big(n^{-1}E\big[\Psi^{dc}_{2121}(i,j;a)^2\big]^{1/2}\big) + O_p\big(n^{-1}E\big[\Psi^{dc}_{2122}(i,j;a)^2\big]^{1/2}\big) \\
&\quad + O_p\big(n^{-1}E\big[\Psi^{dc}_{2123}(i,j;a)^2\big]^{1/2}\big) \\
&= O_p(n^{-1/2}b_1^{\nu_1}) + O_p(n^{-1}b_1^{-1/2}) = o_p(n^{-1/2}).
\end{aligned}
$$

by assumption A5. As a result,

$$
B_{212dc}(a) = o_p(n^{-1/2}) + U^{(2)dc}_{2121}(a) + U^{(2)dc}_{2122}(a) + U^{(2)dc}_{2123}(a) = o_p(n^{-1/2}).
$$

The proof of the order of B_{211d} is, mutatis mutandis, virtually identical to proof of the order of B_{212d}, thus one can conclude that, $B_{211d} = o_p(n^{-1/2})$. Furthermore, by Assumption A3,

$$
B_{21d} = B_{211d} - \sum_{c=1}^{p} \beta_{1c}B_{212dc} \le O_p(n^{-1/2})(1 + ||\beta_1||_E) = o_p(n^{-1/2}).
$$

Consequently, $B_2 = \sum_{d=1}^{D}\big[B_{21d} + B_{22d} + B_{23d} + B_{24d}\big] = o_p(n^{-1/2})$.

$$
\begin{aligned}
B_{3d} &= n^{-1}\big[\mathbf{Z}_n - \hat{\mathbf{H}}^*_n(Z)\big]'\hat{\boldsymbol{\phi}}_n\big[\mathsf{S}^d_{2n}(Y) - \mathsf{S}^d_{2n}(Z)\beta_1\big] \\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i - H^*(Z_i) + H^*(Z_i) - \hat{H}^*(Z_i)\big]\big(\phi_i + \hat{\phi}_i - \phi_i\big)\big[\mathsf{S}^d_2(Y_i) - \mathsf{S}^d_2(Z_i)'\beta_1\big] \\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i - H^*(Z_i)\big]\phi_i\Big(H^d_2(Y_i) - \hat{H}^d_2(Y_i) - \sum_{c=1}^{p}\big[H^d_2(Z_{ci}) - \hat{H}^d_2(Z_{ci})\big]\beta_{1c}\Big) \\
&\quad + n^{-1}\sum_{i=1}^{n}\big[Z_i - H^*(Z_i)\big]\big(\hat{\phi}_i - \phi_i\big)\Big(H^d_2(Y_i) - \hat{H}^d_2(Y_i) - \sum_{c=1}^{p}\big[H^d_2(Z_{ci}) - \hat{H}^d_2(Z_{ci})\big]\beta_{1c}\Big) \\
&\quad + n^{-1}\sum_{i=1}^{n}\big[H^*(Z_i) - \hat{H}^*(Z_i)\big]\phi_i\Big(H^d_2(Y_i) - \hat{H}^d_2(Y_i) - \sum_{c=1}^{p}\big[H^d_2(Z_{ci}) - \hat{H}^d_2(Z_{ci})\big]\beta_{1c}\Big)
\end{aligned}
$$

$$
+ n^{-1}\sum_{i=1}^{n}\big[H^*(Z_i)-\hat{H}^*(Z_i)\big]\big(\hat{\phi}_i-\phi_i\big)\Big(H_2^d(Y_i)-\hat{H}_2^d(Y_i)-\sum_{c=1}^{p}\big[H_2^d(Z_{ci})-\hat{H}_2^d(Z_{ci})\big]\beta_{1c}\Big)
$$

$$
\equiv B_{31d}+B_{32d}+B_{33d}+B_{34d}
$$

$$
\begin{aligned}
B_{32d} &= n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\big[\mathsf{S}_{2n}^d(Y)-\mathsf{S}_{2n}^d(Z)\beta_1\big]\\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\big(\hat{\phi}_i-\phi_i\big)\Big(H_2^d(Y_i)-\hat{H}_2^d(Y_i)-\sum_{c=1}^{p}\big[H_2^d(Z_{ci})-\hat{H}_2^d(Z_{ci})\big]\beta_{1c}\Big)\\
&\le \sup_{XV\in G_{XV}}|\hat{\phi}(X,\hat{V})-\phi(X,V)|\Big[\sup_{V_d\in G_{V_d}}\big|H_2^d(Y)-\hat{H}_2^d(Y)\big|\\
&\qquad + \max_{1\le j\le p}\sup_{V_d\in G_{V_d}}\big|H_2^d(Z_j)-\hat{H}_2^d(Z_j)\big|\sum_{c=1}^{p}|\beta_{1c}|\Big]n^{-1}\sum_{i=1}^{n}\big|Z_i-H^*(Z_i)\big|\\
&\le O_p\big(\mathcal{L}_{0n}\big)O_p\big(\mathcal{L}_{2n}\big)\big(1+||\beta_1||_E\big)O_p(1)=O_p\big(\mathcal{L}_{0n}\mathcal{L}_{2n}\big)=o_p(n^{-1/2}),
\end{aligned}
$$

by Theorems 1 and 2, Lemma 3 xxvi) and xxvii), Assumption A3, and Markov's Inequality.

$$
\begin{aligned}
B_{33d}(c) &= n^{-1}\big[\mathbf{H}_n^*(Z_c)-\hat{\mathbf{H}}_n^*(Z_c)\big]'\boldsymbol{\phi}_n\big[\mathsf{S}_{2n}^d(Y)-\mathsf{S}_{2n}^d(Z)\beta_1\big]\\
&= n^{-1}\sum_{i=1}^{n}\big[H^*(Z_{ci})-\hat{H}^*(Z_{ci})\big]\phi_i\Big(H_2^d(Y_i)-\hat{H}_2^d(Y_i)-\sum_{c=1}^{p}\big[H_2^d(Z_{ci})-\hat{H}_2^d(Z_{ci})\big]\beta_{1c}\Big)\\
&\le \sup_{XV\in G_{XV}}|H^*(Z_c)-\hat{H}^*(Z_c)|\Big[\sup_{V_d\in G_{V_d}}\big|H_2^d(Y)-\hat{H}_2^d(Y)\big|\\
&\qquad + \max_{1\le j\le p}\sup_{V_d\in G_{V_d}}\big|H_2^d(Z_j)-\hat{H}_2^d(Z_j)\big|\sum_{c=1}^{p}|\beta_{1c}|\Big]n^{-1}\sum_{i=1}^{n}\phi_i\\
&\le O_p\big(\mathcal{L}_n\mathcal{L}_{2n}\big)\big(1+||\beta_1||_E\big)O(1)=O_p\big(\mathcal{L}_n\mathcal{L}_{2n}\big)=o_p(n^{-1/2}),
\end{aligned}
$$

by Theorem 2, Lemma 3 xxvi) and xxvii), and Assumption A3.

$$
\begin{aligned}
B_{34d}(c) &= n^{-1}\big[\mathbf{H}_n^*(Z_c)-\hat{\mathbf{H}}_n^*(Z_c)\big]'\big(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n\big)\big[\mathsf{S}_{2n}^d(Y)-\mathsf{S}_{2n}^d(Z)\beta_1\big]\\
&= n^{-1}\sum_{i=1}^{n}\big[H^*(Z_{ci})-\hat{H}^*(Z_{ci})\big]\big(\hat{\phi}_i-\phi_i\big)\Big(H_2^d(Y_i)-\hat{H}_1^d(Y_i)-\sum_{c=1}^{p}\big[H_2^d(Z_{ci})-\hat{H}_2^d(Z_{ci})\big]\beta_{1c}\Big)\\
&\le \sup_{XV\in G_{XV}}|\hat{\phi}(X,\hat{V})-\phi(X,V)|\sup_{XV\in G_{XV}}|H^*(Z_c)-\hat{H}^*(Z_c)|\Big[\sup_{V_d\in G_{V_d}}\big|H_2^d(Y)-\hat{H}_2^d(Y)\big|\\
&\qquad + \max_{1\le j\le p}\sup_{V_d\in G_{V_d}}\big|H_2^d(Z_j)-\hat{H}_2^d(Z_j)\big|\sum_{c=1}^{p}|\beta_{1c}|\Big]\\
&= O_p\big(\mathcal{L}_n\big)O_p\big(\mathcal{L}_{0n}\big)O_p\big(\mathcal{L}_{2n}\big)\big(1+||\beta_1||_E\big)=O_p\big(\mathcal{L}_n\big)O_p\big(\mathcal{L}_{0n}\big)O_p\big(\mathcal{L}_{2n}\big)=o_p(n^{-1/2}),
\end{aligned}
$$

by Theorems 1 and 2, Lemma 3 xxvi) and xxvii), and Assumption A3.

$$
\begin{aligned}
B_{31d} &= n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]\boldsymbol{\phi}_n\big[\mathsf{S}_{2n}^d(Y)-\mathsf{S}_{2n}^d(Z)\beta_1\big]\\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\phi_i\Big(H_2^d(Y_i)-\hat{H}_2^d(Y_i)-\sum_{c=1}^{p}\big[H_2^d(Z_{ci})-\hat{H}_2^d(Z_{ci})\big]\beta_{1c}\Big)\\
&= (-1)n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\phi_i\big(\hat{H}_2^d(Y_i)-H_2^d(Y_i)\big)\\
&\qquad + \sum_{c=1}^{p}\beta_{1c}\,n^{-1}\sum_{i=1}^{n}\big[Z_i-H^*(Z_i)\big]\phi_i\big[\hat{H}_2^d(Z_{ci})-H_2^d(Z_{ci})\big]\\
&= -B_{311d}+\sum_{c=1}^{p}\beta_{1c}B_{312dc}.
\end{aligned}
$$

$$
\begin{aligned}
B_{312dc}(a) &= n^{-1}\sum_{i=1}^{n}\big[Z_{ai}-H^*(Z_{ai})\big]\phi_i\big[\hat{H}_2^d(Z_{ci})-H_2^d(Z_{ci})\big]\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}\Big\{[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(\hat{V}_d)\hat{\theta}_{2j}^d Z_{ci}-H_2^d(Z_{ci})\Big\}\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(\hat{V}_d)\Big[\big(\hat{\theta}_{2j}^d-\theta_{2j}^d\big)Z_{cj}+\theta_{2j}^d\big(Z_{cj}-E[Z_{cj}|X_j,V_j]\big)\\
&\qquad\qquad+\theta_{2j}^d\big(E[Z_{cj}|X_j,V_j]-H_2^d(Z_{cj})\big)+\theta_{2j}^d\big(H_2^d(Z_{cj})-H_2^d(Z_{ci})\big)\Big]\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}[b_2^{-1}(\hat{V}_{dj}-\hat{V}_{di})]\mathsf{C}_{2ji}^d(c)\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[(n-1)b_2]^{-1}\sum_{j\neq i}K_{2ji}(V_d)\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[(n-1)b_2^2]^{-1}\sum_{j\neq i}K_{2ji}^{(1)}(V_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})\big]\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[2(n-1)b_2^3]^{-1}\sum_{j\neq i}K_{2ji}^{(2)}(V_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})\big]^2\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[6(n-1)b_2^4]^{-1}\sum_{j\neq i}K_{2ji}^{(3)}(V_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})\big]^3\mathsf{C}_{2ji}^d(c)\\
&\quad+n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[24(n-1)b_2^5]^{-1}\sum_{j\neq i}K_{2ji}^{(4)}(\tilde{V}_d)[(\hat{V}_{dj}-V_{dj})-(\hat{V}_{di}-V_{di})\big]^4\mathsf{C}_{2ji}^d(c)\\
&= [n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}(V_d)\mathsf{C}_{2ji}^d(c)\\
&\quad-[n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big]\mathsf{C}_{2ji}^d(c)\\
&\quad+[n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)\big]\mathsf{C}_{2ji}^d(c)\\
&\quad+[2n(n-1)b_2^2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(2)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^2\mathsf{C}_{2ji}^d(c)\\
&\quad+[6n(n-1)b_2^3]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(3)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^3\mathsf{C}_{2ji}^d(c)\\
&\quad+[24n(n-1)b_2^5]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}K_{2ji}^{(4)}(\tilde{V}_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^4\mathsf{C}_{2ji}^d(c)\\
&\equiv B_{3121dc}(a)+B_{3122dd}(a)+B_{3123dc}(a)+B_{3124dc}(a)+B_{3125dc}(c)+B_{3126dc}(a).
\end{aligned}
$$

Note that proof of the order of B_{3121dc} is, mutatis mutandis, virtually identical to the proof of the order of B_{212dc}. As a result the arguments are not repeated here and one can conclude that, $B_{3121dc} = o_p(n^{-1/2})$.
Note that by Lemma A2, Assumptions 4, and Assumption A5,

$$
E\big[\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\big]=E\big[K_{2ji}^{(1)}(V_d)E\big(\phi_i\zeta_{ai}|V_{di},S_j\big)\big]=0,
$$

$$E\big[H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\big] = E\big[H_2^d(Z_{ci})K_{2ji}^{(1)}(V_d)E\big(\phi_i\zeta_{ai}|V_{di}, S_j\big)\big] = 0,$$

and,

$$\sup_{V_{di}\in G_{V_d}} E\big[|\phi_i\zeta_{ai}\big|^2 V_{di}\big] = O(1),$$
$$\sup_{V_{di}\in G_{V_d}} E\big[|H_2^d(Z_{ci})\phi_i\zeta_{ai}\big|^2 V_{di}\big] \le \sup_{V_{di}\in G_{V_d}} H_2^d(Z_{ci})^2 \sup_{V_{di}\in G_{V_d}} E\big[|\phi_i^2\zeta_{ai}^2|V_{di}\big] = O(1).$$

Consequently, by a trivial modification of Lemma 6,

$$\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^n\Big[\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d) - E\Big(\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\Big)\Big]\Big| = O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right),$$
$$\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^n\Big[H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d) - E\Big(H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\Big)\Big]\Big| = O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right).$$

Consider,

$$\begin{aligned}
B_{3122dc}(a) &= [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_j) - m_d(W_j)]\,\mathsf{C}_{2ji}^d(c)\\
&= (n-1)^{-1}\sum_{j\neq i}[\hat{m}_d^{l_n}(W_j) - m_d(W_j)]\,\mathsf{C}_{2j}^{d*}[nb_2^2]^{-1}\sum_{i=1}^n\Big[\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d) - E\Big(\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\Big)\Big]\\
&\qquad - (n-1)^{-1}\sum_{j\neq i}[\hat{m}_d^{l_n}(W_j) - m_d(W_j)][nb_2^2]^{-1}\sum_{i=1}^n\Big[H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d) - E\Big(H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\Big)\Big]\\
&\le \sup_{W\in G_W}[\hat{m}_d^{l_n}(W) - m_d(W)]\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^n\Big[\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d) - E\Big(\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\Big)\Big]\Big|(n-1)^{-1}\sum_{j\neq i}|\mathsf{C}_{2j}^{d*}|\\
&\qquad + \sup_{W\in G_W}[\hat{m}_d^{l_n}(W) - m_d(W)]\sup_{V_{dj}\in G_{V_d}}\Big|[nb_2^2]^{-1}\sum_{i=1}^n\Big[H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d) - E\Big(H_2^d(Z_{ci})\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\Big)\Big]\Big|\\
&= O_p(L_n)O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right)\sum_{j\neq i}|\mathsf{C}_{2j}^{d*}| + O_p(L_n)O_p\left(\left[\frac{\log(n)}{nb_2^3}\right]^{1/2}\right)\\
&\le O_p(L_nb_2^{-1})O_p\left(\left[\frac{\log(n)}{nb_2}\right]^{1/2}\right)\Big\{1 + \sup_{XV\in G_{XV}}|\hat{\theta}_{2j}^d(X,\hat{V}) - \theta_{2j}^d(X,V)|n^{-1}\sum_{j=1}^n|Z_{cj}|\\
&\qquad + n^{-1}\sum_{j=1}^n|\theta_{2j}^d\rho_{cj}| + n^{-1}\sum_{j=1}^n|\theta_{2j}^d\eta_{2cj}^d| + n^{-1}\sum_{j=1}^n|\theta_{2j}^dH_2^d(Z_{cj})|\Big\}\\
&= o_p(n^{-1/2})O_p(1) = o_p(n^{-1/2}),
\end{aligned}$$

by Theorem 1, Lemmas 3 vi) and xxiv) and 6, and Markov's Inequality.

$$B_{3123dc}(a) = [n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)]\,\mathsf{C}_{2ji}^d(c).$$

Recall that by assumption A2,

$$\lim_{\gamma\to\infty}K_2^{(1)}(\gamma)\gamma = 0, \quad \text{and} \quad \lim_{\gamma\to-\infty}K_2^{(1)}(\gamma)\gamma = 0.$$

Thus,

$$E\big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})]\big|S_{-j}\big]$$

$$= \phi_i \zeta_{ai} H_2^{(1)d}(Z_{ci}) \int b_2^{-1} K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj} - V_{di})] g(X_j, V_{-dj}) p(X_j, V_j)^{-1} p(X_j, V_j) dX_j dV_j$$
$$= \phi_i \zeta_{ai} H_2^{(1)d}(Z_{ci}) \int b_2^{-1} K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj} - V_{di})] dV_{dj} \int g(X_j, V_{-dj}) dV_{-dj} dX_j$$
$$= \phi_i \zeta_{ai} H_2^{(1)d}(Z_{ci}) \int b_2^{-1} K_2^{(1)}(\gamma) b_2 d\gamma$$
$$= \phi_i \zeta_{ai} H_2^{(1)d}(Z_{ci}) \Big[\lim_{\gamma \to \infty} K_2^{(1)}(\gamma)\gamma - \lim_{\gamma \to -\infty} K_2^{(1)}(\gamma)\gamma - \int K_2(\gamma) d\gamma \Big]$$
$$= -\phi_i \zeta_{ai} H_2^{(1)d}(Z_{ci}).$$

Eliminating the $E[Z_{cj}|X_j, V_j]$ terms from $\mathsf{C}_{2ji}^d(c)$ one has,

$$\begin{aligned}
B_{3123dc}(a) &= [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} b_2^{-1} K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)] \, \mathsf{C}_{2ji}^d(c) \\
&= [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} b_2^{-1} K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)] \big(\hat{\theta}_{2j}^d - \theta_{2j}^d\big) Z_{cj} \\
&\quad + [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} b_2^{-1} K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)] \theta_{2j}^d \big(Z_{cj} - H_2^d(Z_{cj})\big) \\
&\quad + [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} b_2^{-1} K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)] \theta_{2j}^d \big(H_2^d(Z_{cj}) - H_2^d(Z_{ci})\big) \\
&\equiv B_{31231dc}(a) + B_{31232dc}(a) + B_{31233dc}(a).
\end{aligned}$$

$$\begin{aligned}
B_{31231dc}(a) &= [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i}^n \phi_i \zeta_{ai} b_2^{-1} K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i) - m_d(W_i)] \big(\hat{\theta}_{2j}^d - \theta_{2j}^d\big) Z_{cj} \\
&\leq \sup_{XV \in G_{X,V}} \big|\hat{\theta}_2^d(X, \hat{V}) - \theta_2^d(X, V)\big| \sup_{W \in G_W} |\hat{m}_d^{l_n}(W) - m_d(W)| \\
&\qquad \times \sup_{V_{di} \in G_{V_d}} [(n-1)b_2]^{-1} \sum_{j \neq i} |b_2^{-1} K_{2ji}^{(1)}(V_d)| n^{-1} \sum_{i=1}^n |\phi_i \zeta_{ai}| \\
&= O_p\big(\mathcal{L}_{0n}\big) O_p(L_n) O_p(b_2^{-1}) O_p(1) = O_p\big(\mathcal{L}_{0n} L_n b_2^{-1}\big) = o_p(n^{-1/2}).
\end{aligned}$$

by Theorem 1 and Lemma 3. Note that,

$$\begin{aligned}
E\big[K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big(Z_{cj} - H_2^d(Z_{cj})\big)\big] &= E\big[K_{2ji}^{(1)}(V_d) p(V_{dj})^{-1} \phi_j \big(Z_{cj} - H_2^d(Z_{cj})\big)\big] \\
&= E\big[K_{2ji}^{(1)}(V_d) p(V_{dj})^{-1} E\big(\phi_j Z_{cj} - \phi_j H_2^d(Z_{cj}) \big| V_{dj}, V_{di}\big)\big] \\
&= E\big[K_{2ji}^{(1)}(V_d) p(V_{dj})^{-1} \big(H_d^2(Z_{cj}) - H_2^d(Z_{cj}) E[\phi_j | V_{dj}, V_{di}]\big)\big] \\
&= E\big[K_{2ji}^{(1)}(V_d) p(V_{dj})^{-1} H_d^2(Z_{cj}) \big(1 - \int g(X_j, V_j) p(X_j, V_j)^{-1} p(X_j, V_j) p(V_{dj})^{-1} dX_j, dV_{-dj}\big)\big] \\
&= E\big[K_{2ji}^{(1)}(V_d) p(V_{dj})^{-1} H_d^2(Z_{cj}) \big(1 - \int g(X_j, V_{-dj}) dX_j, dV_{-dj}\big)\big] \\
&= 0,
\end{aligned}$$

By Assumption A5, $E\big(|Z_{cj} - H_2^d(Z_{cj})|^2 \big| V_{dj}\big) = O(1)$. Then by Lemma 6,

$$\sup_{V_{di} \in G_{V_d}} \Big| [(n-1)b_2^2]^{-1} \sum_{j \neq i} \Big[K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big[Z_{cj} - H_2^d(Z_{cj})\big] - E\Big(K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big[Z_{cj} - H_2^d(Z_{cj})\big]\Big)\Big] \Big| = O_p\left(\left[\frac{\log(n)}{(n-1)b_2^3}\right]^{1/2}\right).$$

Consider,

$$
\begin{aligned}
B_{31232dc}(a) &= [n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)]\theta_{2j}^d\big(Z_{cj}-H_2^d(Z_{cj})\big)\\
&= n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ai}[\hat{m}_d^{l_n}(W_i)-m_d(W_i)][(n-1)b_2^2]^{-1}\sum_{j\neq i}K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big[Z_{cj}-H_2^d(Z_{cj})\big]\\
&\leq \sup_{V_{di}\in G_{V_d}}\Big|[(n-1)b_2^2]^{-1}\sum_{j\neq i}\Big[K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big[Z_{cj}-H_2^d(Z_{cj})\big]-E\Big(K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big[Z_{cj}-H_2^d(Z_{cj})\big]\Big)\Big]\Big|\\
&\qquad\times\sup_{W\in G_W}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|n^{-1}\sum_{i=1}^{n}|\phi_i\zeta_{ai}|\\
&= O_p\left(\left[\frac{\log(n)}{(n-1)b_2^3}\right]^{1/2}\right)O_p(L_n)O_p(1)=O_p(L_nb_2^{-1})O_p\left(\left[\frac{\log(n)}{(n-1)b_2}\right]^{1/2}\right)=o_p(n^{-1/2}),
\end{aligned}
$$

by Lemmas 3 x) and xxvii) and 6, and Markov's Inequality.

$$
\begin{aligned}
B_{31233dc}(a) &= [n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)]\theta_{2j}^d\big(H_2^d(Z_{cj})-H_2^d(Z_{ci})\big)\\
&= n^{-1}\sum_{i=1}\phi_i\zeta_{ai}[\hat{m}_d^{l_n}(W_i)-m_d(W_i)][(n-1)b_2^2]^{-1}\sum_{j\neq i}K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big(H_2^d(Z_{cj})-H_2^d(Z_{ci})\big).
\end{aligned}
$$

For some $\lambda\in(0,1)$, let $H_2^{(\nu_2)d}(\tilde{Z}_{ci})=\frac{d^{\nu_2}}{d^{\nu_2}V_{dj}}E[\phi_jZ_{cj}|V_{dj}]\Big|_{V_{dj}=\lambda V_{dj}+(1-\lambda)V_{di}}$. Consider,

$$
\begin{aligned}
&[(n-1)b_2^2]^{-1}\sum_{j\neq i}K_{2ji}^{(1)}(V_d)\theta_{2j}^d\big(H_2^d(Z_{cj})-H_2^d(Z_{ci})\big)\\
&\quad= [(n-1)b_2^2]^{-1}\sum_{j\neq i}K_{2ji}^{(1)}(V_d)\theta_{2j}^d\left(\sum_{m=1}^{\nu_2-1}(m!)^{-1}H_2^{(m)d}(Z_{ci})(V_{dj}-V_{di})^m+(\nu_2!)^{-1}H_2^{(\nu_2)d}(\tilde{Z}_{ci})(V_{dj}-V_{di})^{\nu_2}\right)\\
&\quad= \sum_{m=1}^{\nu_2-1}(m!)^{-1}H_2^{(m)d}(Z_{ci})[(n-1)b_2^{1-m}]^{-1}\sum_{j\neq i}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m\\
&\qquad\quad+(\nu_2!)^{-1}[(n-1)b_2^{1-\nu_2}]^{-1}\sum_{j\neq i}H_2^{(\nu_2)d}(\tilde{Z}_{ci})b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^{\nu_2}\\
&\quad\equiv \sum_{m=1}^{\nu_2-1}(m!)^{-1}A_{di}^m(c)+(\nu_2!)^{-1}A_{di}^{\nu_2}(c).
\end{aligned}
$$

$$
A_{di}^m(c)=H_2^{(m)d}(Z_{ci})[(n-1)b_2^{1-m}]^{-1}\sum_{j\neq i}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m
$$

Note that for $2\leq m\leq\nu_2-1$, one has by Assumption A2,

$$
\begin{aligned}
&E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)^2\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^{2m}\Big|V_{di}\Big)\\
&\quad= \int b_2^{-1}K_{2ji}^{(1)}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]^{2m}g(X_j,V_{-dj})p(X_j,V_j)^{-1}p(X_j,V_j)dX_jdV_j\\
&\quad= \int b_2^{-1}K_{2ji}^{(1)}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]^{2m}dV_{dj}\int g(X_j,V_{-dj})dX_jdV_j\\
&\quad= \int b_2^{-1}K_2^{(1)}(\gamma)^2\gamma^{2m}b_2d\gamma\int K_2^{(1)}(\gamma)^2\gamma^{2m}d\gamma=O(1),
\end{aligned}
$$

and,

$$
E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m\Big|V_{di}\Big)
$$

$$
\begin{aligned}
&= \int b_2^{-1} K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj}-V_{di})]^m g(X_j,V_{-dj})p(X_j,V_j)^{-1}p(X_j,V_j)dX_j dV_j \\
&= \int b_2^{-1} K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj}-V_{di})]^m dV_{dj} \int g(X_j,V_{-dj})dX_j dV_j \\
&= \int b_2^{-1} K_2^{(1)}(\gamma)\gamma^m b_2 d\gamma = \int K_2^{(1)}(\gamma)\gamma^m d\gamma \\
&= \lim_{\gamma\to\infty} K_2(\gamma)\gamma^m - \lim_{\gamma\to-\infty} K_2(\gamma)\gamma^m - m\int K_2(\gamma)\gamma^{m-1}d\gamma = 0.
\end{aligned}
$$

Recall that $m \geq 2$ and note that,

$$
\begin{aligned}
&E\Big[\Big([(n-1)b_2^{1-m}]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m\Big)^2\Big] \\
&\quad \leq \sup_{X,V\in G_{X,V}} \big|\theta_2^d(X,V)\big|[(n-1)^2 b_2^{3-2m}]^{-1}\sum_{j\neq i} E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)^2\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^{2m}\Big) \\
&\qquad + [(n-1)^2 b_2^{2-2m}]^{-1}\sum_{j\neq i}\sum_{\substack{g\neq i\\ g\neq j}} E\Big[E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m\Big|V_{di}\Big)^2\Big] \\
&\quad = O\big([(n-1)b_2^{3-2m}]^{-1}\big) \leq O\big(b_2(n-1)^{-1}\big).
\end{aligned}
$$

Furthermore by Markov's Inequality,

$$
[(n-1)b_2^{1-m}]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m = O_p(b_2^{1/2}(n-1)^{-1/2}) = o_p(n^{-1/2}).
$$

and by Assumption A4

$$
\begin{aligned}
A_{dc}^m &= H_2^{(m)d}(Z_{ci})[(n-1)b_2^{1-m}]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m \\
&\leq \sup_{V_{dj}\in G_{V_d}} \big|H_2^{(m)d}(Z_{ci})\big|\Big|[(n-1)b_2^{1-m}]^{-1}\sum_{j\neq i} b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^m\Big| \\
&= O(1)o_p(n^{-1/2}) = o_p(n^{-1/2}).
\end{aligned}
$$

Note that,

$$
\begin{aligned}
&E\Big(\big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\theta_{2j}^d\big|b_2^{-1}(V_{dj}-V_{di})\big|^{\nu_2}\Big|S_i\Big) \\
&\quad = \int \big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\big|b_2^{-1}(V_{dj}-V_{di})\big|^{\nu_2} g(X_j,V_{-dj})p(X_j,V_j)^{-1}p(X_j,V_j)dX_j dV_j \\
&\quad = \int \big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\big|b_2^{-1}(V_{dj}-V_{di})\big|^{\nu_2} dV_{dj}\int g(X_j,V_{-dj})dX_j dV_{-dj} \\
&\quad = b_2^{-1}\int \big|K_2^{(1)}(\gamma)\big|\big|\gamma\big|^{\nu_2} b_2 d\gamma = O(1).
\end{aligned}
$$

Consequently by Markov's Inequality,

$$
\big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\theta_{2j}^d\big|b_2^{-1}(V_{dj}-V_{di})\big|^{\nu_2} = O_p(1).
$$

As a result by Assumption A4,

$$
\begin{aligned}
A_{di}^{\nu_2}(c) &= [(n-1)b_2^{1-\nu_2}]^{-1}\sum_{j\neq i} H_2^{(\nu_2)d}(\tilde{Z}_{ci})b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d(V_{dj}-V_{di})^{\nu_2} \\
&\leq \sup_{V_{dj},V_{di}\in G_{V_d}} \big|H_2^{(\nu_2)d}(\tilde{Z}_{ci})\big|[(n-1)b_2^{1-\nu_2}]^{-1}\sum_{j\neq i}\big|b_2^{-1}K_{2ji}^{(1)}(V_d)\big|\theta_{2j}^d\big|b_2^{-1}(V_{dj}-V_{di})\big|^{\nu_2}
\end{aligned}
$$

$$= O_p(b_2^{\nu_2 - 1}).$$

Consequently,

$$[(n-1)b_2^2]^{-1} \sum_{j \neq i} K_{2ji}^{(1)}(V_d)\theta_{2j}^d \big(H_2^d(Z_{cj}) - H_2^d(Z_{ci})\big)$$
$$= H_2^{(1)d}(Z_{ci})[n-1]^{-1} \sum_{j \neq i} b_2^{-1} K_{2ji}^{(1)}(V_d)\theta_{2j}^d [b_2^{-1}(V_{dj} - V_{di})] + O_p(b_2^{\nu_2 - 1}) + o_p(n^{-1/2}).$$

Furthermore by Lemma 3 vi) and xxiii) and Markov's Inequality,

$$\begin{aligned}
B_{31233dc}(a) &= n^{-1} \sum_{i=1}^n \phi_i \zeta_{ai} [\hat{m}_d^{l_n}(W_i) - m_d(W_i)] [(n-1)b_2^2]^{-1} \sum_{j \neq i} K_{2ji}^{(1)}(V_d)\theta_{2j}^d \big(H_2^d(Z_{cj}) - H_2^d(Z_{ci})\big) \\
&\leq n^{-1} \sum_{i=1}^n \phi_i \zeta_{ai} [\hat{m}_d^{l_n}(W_i) - m_d(W_i)] H_2^{(1)d}(Z_{ci})[(n-1)b_2]^{-1} \sum_{j \neq i} K_{2ji}^{(1)}(V_d)\theta_{2j}^d [b_2^{-1}(V_{dj} - V_{di})] \\
&\quad + \sup_{W \in G_W} |\hat{m}_d^{l_n}(W) - m_d(W)|] \big(O_p(b_2^{\nu_2 - 1}) + o_p(n^{-1/2})\big) n^{-1} \sum_{i=1}^n |\phi_i \zeta_{ci}| \\
&= [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})][\hat{m}_d^{l_n}(W_i) - m_d(W_i)] \\
&\quad + O_p(L_n b_2^{-1}) O_p(b_2^{\nu_2}) + O_p(L_n) o_p(n^{-1/2}) \\
&\equiv B_{31233dc}^*(a) + o_p(n^{-1/2}).
\end{aligned}$$

$$B^*{}_{31233dc}(a) = [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})][m_d(W_i) - \hat{m}_d^{l_n}(W_i)]$$

$$\begin{aligned}
&= [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ai})[b_2^{-1}(V_{dj} - V_{di})][\hat{m}_d^{l_n}(W_i) - m_d^{l_n}(W_i)] \\
&\quad + [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})][m_d^{l_n}(W_i) - m_d(W_i)] \\
&= [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})][m_d^{l_n}(W_i) - m_d(W_i)] \\
&\quad - [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})] \mathbf{B}_n(W_i)' Q_{BB}^{-1} n^{-1} \mathbf{B}_n' \mathbf{V}_d \\
&\quad + [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})] \mathbf{B}_n(W_i)' Q_{BB}^{-1} n^{-1} \mathbf{B}_n' \big(\mathbf{M}_d^{l_n} - \mathbf{M}_d\big) \\
&\quad - [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})] \mathbf{B}_n(W_i)' \big(Q_{nBB}^{-1} - Q_{BB}^{-1}\big) n^{-1} \mathbf{B}_n' \mathbf{V}_d \\
&\quad + [n(n-1)b_2]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})] \mathbf{B}_n(W_i)' \big(Q_{nBB}^{-1} - Q_{BB}^{-1}\big) n^{-1} \mathbf{B}_n' \big(\mathbf{M}_d^{l_n} - \mathbf{M}_d\big) \\
&\equiv R_{1dc}(a) - R_{2dc}(a) + R_{3dc}(a) - R_{4dc}(a) + R_{5dc}(a).
\end{aligned}$$

$$R_{1dc}(a) = [n(n-1)]^{-1} \sum_{i=1}^n \sum_{j \neq i} \phi_i \zeta_{ai} b_2^{-1} K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj} - V_{di})][m_d^{l_n}(W_i) - m_d(W_i)]$$

$$= [n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_1^d(i,j;a,c) \simeq \binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_1^d(i,j;a,c) = U_1^{(2)d}(a,c).$$

where $\Gamma_1^d(i,j;a,c) = \Psi_1^d(i,j;a,c) + \Psi_1^d(j,i;a,c)$.

$$\begin{aligned}
E\big(\Psi_1^d(i,j;a,c)\big) &= E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})][m_d^{l_n}(W_i)-m_d(W_i)]\Big)\\
&= E\Big(\phi_i\zeta_{ai}\theta_{2j}^d H_2^{(1)d}(Z_{ci})[m_d^{l_n}(W_i)-m_d(W_i)]E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj}-V_{di})]\Big|S_i\Big]\Big)\\
&\leq \sup_{W\in G_W}|m_d^{l_n}(W)-m_d(W)|\sup_{V_d\in G_{V_d}}\big|H_2^{(1)d}(Z_c)\big|E(|\phi_i\zeta_{ai}|)E(\theta_{2j}^d) = O(l_n^{-k}).
\end{aligned}$$

$$\begin{aligned}
E\big[E\big(\Psi_1^d(i,j;a,c)\big|S_i\big)^2\big]^{1/2} &= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})][m_d^{l_n}(W_i)-m_d(W_i)]\Big|S_i\Big)^2\Big]^{1/2}\\
&= E\Big[\phi_i^2\zeta_{ai}^2[m_d^{l_n}(W_i)-m_d(W_i)]^2 H_2^{(1)d}(Z_{ci})^2 E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj}-V_{di})]\Big|S_i\Big)^2\Big]^{1/2}\\
&\leq \sup_{W\in G_W}|m_d^{l_n}(W)-m_d(W)|\sup_{V_d\in G_{V_d}}\big|H_2^{(1)d}(Z_c)\big|E\big[\phi_i^2\zeta_{ai}^2\big]^{1/2} = O(l_n^{-k}).
\end{aligned}$$

$$\begin{aligned}
E\big[E\big(\Psi_1^d(i,j;a,c)\big|S_j\big)^2\big]^{1/2} &= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})][m_d^{l_n}(W_i)-m_d(W_i)]\Big|S_j\Big)^2\Big]^{1/2}\\
&\leq \sup_{W\in G_W}|m_d^{l_n}(W)-m_d(W)|\sup_{V_d\in G_{V_d}}\big|H_2^{(1)d}(Z_c)\big|\sup_{XV\in G_{XV}}\theta_2^d(X,V)\\
&\qquad\times E\Big[E\Big(E\big[|\phi_i\zeta_{ai}|\big|V_{di},S_j\big]b_2^{-1}|K_{2ji}^{(1)}(V_d)||b_2^{-1}(V_{dj}-V_{di})|\Big|S_j\Big)^2\Big]^{1/2}\\
&\leq O(l_n^{-k})E\Big[E\Big(b_2^{-1}|K_{2ji}^{(1)}(V_d)||b_2^{-1}(V_{dj}-V_{di})|\Big|S_j\Big)^2\Big]^{1/2} = O(l_n^{-k}).
\end{aligned}$$

$$\begin{aligned}
E\big(\Psi_1^d(i,j;a,c)^2\big)^{1/2} &= E\Big(\phi_i^2\zeta_{ai}^2 b_2^{-2}K_{2ji}^{(1)}(V_d)^2[\theta_{2j}^d]^2 H_2^{(1)d}(Z_{ci})^2[b_2^{-1}(V_{dj}-V_{di})]^2[m_d^{l_n}(W_i)-m_d(W_i)]^2\Big)^{1/2}\\
&\leq \sup_{W\in G_W}|m_d^{l_n}(W)-m_d(W)|\sup_{V_d\in G_{V_d}}\big|H_2^{(1)d}(Z_c)\big|\sup_{XV\in G_{XV}}\theta_2^d(X,V)\\
&\qquad\times E\Big(\phi_i^2\zeta_{ai}^2 E\Big[b_2^{-2}K_{2ji}^{(1)}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]^2\Big|S_i\Big]\Big)^{1/2}\\
&= O(l_n^{-k}b_2^{-1/2})E\big(\phi_i^2\zeta_{ai}^2\big)^{1/2} = O(l_n^{-k}b_2^{-1/2}).
\end{aligned}$$

Consequently by Assumption A6,

$$\begin{aligned}
R_{1dc}(a) \simeq U_1^{(2)d}(a,c) &= E\big(\Psi_1^d(i,j;a,c)\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi_1^d(i,j;a,c)\big|S_i\big)^2\big]^{1/2}\big)\\
&\quad + O_p\big(n^{-1/2}E\big[E\big(\Psi_1^d(i,j;a,c)\big|S_j\big)^2\big]^{1/2}\big) + O_p\big(n^{-1}E\big(\Psi_1^d(i,j;a,c)^2\big)^{1/2}\big)\\
&= O(l_n^{-k}) + O_p(n^{-1/2}l_n^{-k}) + O_p\big(n^{-1}O(l_n^{-k}b_2^{-1/2})\big) = o_p(n^{-1/2}).
\end{aligned}$$

Note that,

$$\begin{aligned}
&E\Big(||(n-1)^{-1}\theta_{2j}^d b_2^{-1}\mathbf{K}_{2j}^{(1)d}(V_d)'\boldsymbol{\phi}_i\dot{\zeta}_c\dot{\mathbf{H}}_2^{(1)d}(Z_{ci})[b_2^{-1}(\dot{\mathbf{V}}_{dj}-\dot{\mathbf{V}}_d)]\mathbf{B}_n||_E^2\Big)\\
&\quad= E\Big(||(n-1)^{-1}\sum_{i\neq j}\mathbf{B}_n(W_i)\phi_i\zeta_{ci}H_2^{(1)d}(Z_{ci})b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]||_E^2\Big)\\
&\quad= E\Big([\theta_{2j}^d]^2 b_2^{-1}(n-1)^{-2}\sum_{i\neq j}\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)[\phi_i\zeta_{ci}]^2 H_2^{(1)d}(Z_{ci})b_2^{-1}K_{2ji}^{(1)}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]^2\Big)
\end{aligned}$$

$$+ E\Big([\theta^d_{2j}]^2(n-1)^{-2}\sum_{i\neq j}\sum_{\substack{g\neq j\\ g\neq i}}\mathbf{B}_n(W_i)'\phi_i\zeta_{ci}H_2^{(1)d}(Z_{ci})b_2^{-1}K^{(1)}_{2ji}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]$$

$$\times\,\mathbf{B}_n(W_g)\phi_g\zeta_{cg}H_2^{(1)d}(Z_{cg})b_2^{-1}K^{(1)}_{2jg}(V_d)^2[b_2^{-1}(V_{dj}-V_{dg})]\Big)$$

$$\leq \sup_{X,V\in G_{XV}}\theta^d_2(X,V)^2\sup_{V_d,W\in G_{V_d,W}}E[\phi^2\zeta_c^2|V_d,W]\sup_{V_d\in G_{V_d}}H_2^{(1)d}(Z_c)^2$$

$$\times\Bigg\{b_2^{-1}(n-1)^{-2}\sum_{i\neq j}E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)E\Big[b_2^{-1}K^{(1)}_{2ji}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]^2\Big|W_i\Big]\Big)$$

$$+(n-1)^{-2}\sum_{i\neq j}\sum_{\substack{g\neq j\\ g\neq i}}E\Big(\big|\mathbf{B}_n(W_i)'\big|E\Big[b_2^{-1}K^{(1)}_{2ji}(V_d)^2[b_2^{-1}(V_{dj}-V_{di})]\Big|S_{-i},W_i\Big]\Bigg\}$$

$$\times\,\big|\mathbf{B}_n(W_g)\big|E\Big[b_2^{-1}K^{(1)}_{2jg}(V_d)^2[b_2^{-1}(V_{dj}-V_{dg})]\Big|S_{-g},W_g\Big]\Big)$$

$$= O\big([(n-1)b_2]^{-1}\big)E\Big(\mathbf{B}_n(W_i)'\mathbf{B}_n(W_i)\Big)+E\Big(\big|\mathbf{B}_n(W_i)'\big|\big|\mathbf{B}_n(W_g)\big|\Big)=O\big(l_n[(n-1)b_2]^{-1}\big)+O(1)=O(1).$$

Consequently by Assumption 1, and Markov's inequality,

$$||(n-1)^{-1}\theta^d_{2j}b_2^{-1}\mathbf{K}^{(1)d}_{2j}(V_d)'\boldsymbol{\phi}_i\dot{\zeta}_c\dot{\mathbf{H}}_2^{(1)d}(Z_{ci})[b_2^{-1}(\dot{\mathbf{V}}_{dj}-\dot{\mathbf{V}}_d)]\mathbf{B}_n||_E=O_p(1).$$

Consider,

$$R_{3dc}(a)-R_{4dc}(a)+R_{5dc}(a)$$

$$=[n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i\zeta_{ai}K^{(1)}_{2ji}(V_d)\theta^d_{2j}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]\mathbf{B}_n(W_i)'Q^{-1}_{BB}n^{-1}\mathbf{B}_n'\big(\mathbf{M}_d^{l_n}-\mathbf{M}_d\big)$$

$$-[n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i\zeta_{ai}K^{(1)}_{2ji}(V_d)\theta^d_{2j}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]\mathbf{B}_n(W_i)'\big(Q^{-1}_{nBB}-Q^{-1}_{BB}\big)n^{-1}\mathbf{B}_n'\mathbf{V}_d$$

$$+[n(n-1)b_2]^{-1}\sum_{i=1}^n\sum_{j\neq i}\phi_i\zeta_{ai}K^{(1)}_{2ji}(V_d)\theta^d_{2j}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]\mathbf{B}_n(W_i)'\big(Q^{-1}_{nBB}-Q^{-1}_{BB}\big)n^{-1}\mathbf{B}_n'\big(\mathbf{M}_d^{l_n}-\mathbf{M}_d\big)$$

$$\leq n^{-1}\sum_{j=1}^n\Big\{||(n-1)^{-1}\theta^d_{2j}b_2^{-1}\mathbf{K}^{(1)d}_{2j}(V_d)'\boldsymbol{\phi}_i\dot{\zeta}_c\dot{\mathbf{H}}_2^{(1)d}(Z_{ci})[b_2^{-1}(\dot{\mathbf{V}}_{dj}-\dot{\mathbf{V}}_d)]\mathbf{B}_n||_E||Q^{-1}_{BB}||_{sp}||n^{-1}\mathbf{B}_n'\big(\mathbf{M}_d^{l_n}-\mathbf{M}_d\big)||_E$$

$$+||(n-1)^{-1}\theta^d_{2j}b_2^{-1}\mathbf{K}^{(1)d}_{2j}(V_d)'\boldsymbol{\phi}_i\dot{\zeta}_c\dot{\mathbf{H}}_2^{(1)d}(Z_{ci})[b_2^{-1}(\dot{\mathbf{V}}_{dj}-\dot{\mathbf{V}}_d)]\mathbf{B}_n||_E||Q^{-1}_{nBB}-Q^{-1}_{BB}||_{sp}||n^{-1}\mathbf{B}_n'\mathbf{V}_d||_E$$

$$+||(n-1)^{-1}\theta^d_{2j}b_2^{-1}\mathbf{K}^{(1)d}_{2j}(V_d)'\boldsymbol{\phi}_i\dot{\zeta}_c\dot{\mathbf{H}}_2^{(1)d}(Z_{ci})[b_2^{-1}(\dot{\mathbf{V}}_{dj}-\dot{\mathbf{V}}_d)]\mathbf{B}_n||_E||Q^{-1}_{nBB}-Q^{-1}_{BB}||_{sp}||n^{-1}\mathbf{B}_n'\big(\mathbf{M}_d^{l_n}-\mathbf{M}_d\big)||_E\Big\}$$

$$=n^{-1}\sum_{j=1}^n\Big\{O(1)O(1)O_p(l_n^{-k})+O(1)O_p\left(\frac{l_n}{\sqrt{n}}\right)O_p\left(\left[\frac{l_n}{n}\right]^{1/2}\right)+O(1)O_p\left(\frac{l_n}{\sqrt{n}}\right)O_p(l_n^{-k})\Big\}$$

$$=o_p(n^{-1/2}).$$

Note that,

$$\mathbf{B}_n(W_i)Q^{-1}_{BB}n^{-1}\mathbf{B}_n'\mathbf{V}_d$$

$$=\sum_{e=1}^q\sum_{g=1}^q\sum_{J=1}^{l_n+2k}\sum_{L=1}^{l_n+2k}Q^{-1}_{BB}\big([e-1](l_n+2k)+L,[g-1](l_n+2k)+J\big)B_L(W_{ei})n^{-1}\sum_{t=1}^n B_J(W_{gt})V_{dt}$$

$$=\sum_{e=1}^q\sum_{g=1}^q\sum_{J=1}^{l_n+2k}\sum_{L=1}^{l_n+2k}Q^{-1}_{BB}\big([e-1](l_n+2k)+L,[g-1](l_n+2k)+J\big)B_L(W_{ei})n^{-1}B_J(W_{gi})V_{di}$$

$$+\sum_{e=1}^{q}\sum_{g=1}^{q}\sum_{J=1}^{l_n+2k}\sum_{L=1}^{l_n+2k} Q_{BB}^{-1}\big([e-1](l_n+2k)+L,[g-1](l_n+2k)+J\big)B_L(W_{ei})n^{-1}B_J(W_{gj})V_{dj}$$

$$+\sum_{e=1}^{q}\sum_{g=1}^{q}\sum_{J=1}^{l_n+2k}\sum_{L=1}^{l_n+2k} Q_{BB}^{-1}\big([e-1](l_n+2k)+L,[g-1](l_n+2k)+J\big)B_L(W_{ei})n^{-1}\sum_{\substack{t\neq i\\ t\neq j}} B_J(W_{gt})V_{dt}$$

adopt the following notation,

$$\sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)\equiv \sum_{e=1}^{q}\sum_{g=1}^{q}\sum_{J=1}^{l_n+2k}\sum_{L=1}^{l_n+2k} Q_{BB}^{-1}\big([e-1](l_n+2k)+L,[g-1](l_n+2k)+J\big).$$

Consequently,

$$\begin{aligned}\mathbf{B}_n(W_i)Q_{BB}^{-1}n^{-1}\mathbf{B}_n'\mathbf{V}_d &= \sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)B_L(W_{ei})n^{-1}B_J(W_{gi})V_{di}\\ &\quad+\sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)B_L(W_{ei})n^{-1}B_J(W_{gj})V_{dj}\\ &\quad+\sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)B_L(W_{ei})n^{-1}\sum_{\substack{t\neq i\\ t\neq j}} B_J(W_{gt})V_{dt}.\end{aligned}$$

As a result,

$$\begin{aligned}R_{2dc}(a) &= [n(n-1)b_2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]\mathbf{B}_n(W_i)'Q_{BB}^{-1}n^{-1}\mathbf{B}_n'\mathbf{V}_d\\ &= \sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)\\ &\quad\times\Big\{[n^2(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gi})V_{di}\\ &\qquad+[n^2(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gj})V_{dj}\\ &\qquad+[n^2(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{\substack{t\neq i\\ t\neq j}}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\Big\}\\ &= \sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)\Big\{[n^2(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_{21}^{d}(i,j;a,c,e,g,J,L)\\ &\qquad+[n^2(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\Psi_{22}^{d}(i,j;a,c,e,g,J,L)\\ &\qquad+[n^2(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{\substack{t\neq i\\ t\neq j}}\Psi_{23}^{d}(i,j,t;a,c,e,g,J,L)\Big\}\\ &\simeq \sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)\Big\{n^{-1}\binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_{21}^{d}(i,j;a,c,e,g,J,L)\\ &\qquad+n^{-1}\binom{n}{2}^{-1}\sum_{i=1}^{n}\sum_{i<j}\Gamma_{22}^{d}(i,j;a,c,e,g,J,L)\end{aligned}$$

$$+\binom{n}{3}^{-1}\sum_{i=1}^{n}\sum_{i<j}\sum_{i<j<t}\Gamma_{23}^{d}(i,j,t;a,c,e,g,J,L)\Big\}$$

$$=\sum_{e,g}^{q}\sum_{L,J}^{l_n+2k}Q_{BB}^{-1}(e,g,L,J)\Big\{n^{-1}U_{21}^{(2)d}(a,c,e,g,J,L)+n^{-1}U_{22}^{(2)d}(a,c,e,g,J,L)+U_{23}^{(3)d}(a,c,e,g,J,L)\Big\}.$$

where $\Psi_{21}^{d}(i,j;a,c,e,g,J,L)=\Psi_{21}^{d}(i,j;a,c,e,g,J,L)+\Psi_{21}^{d}(j,i;a,c,e,g,J,L)$, $\Psi_{22}^{d}(i,j;a,c,e,g,J,L)=\Psi_{22}^{d}(i,j;a,c,e,g,J,L)+\Psi_{22}^{d}(j,i;a,c,e,g,J,L)$ and $\Gamma_{23}^{d}(i,j,t;a,c,e,g,J,L)=\Psi_{23}^{d}(i,j,t;a,c,e,g,J,L)+\Psi_{23}^{d}(i,t,j;a,c,e,g,J,L)+\Psi_{23}^{d}(j,i,t;a,c,e,g,J,L)+\Psi_{23}^{d}(j,t,i;a,c,e,g,J,L)+\Psi_{23}^{d}(t,i,j;a,c,e,g,J,L)+\Psi_{23}^{d}(t,j,i;a,c,e,g,J,L)$

$$\begin{aligned}
&E\big(\Psi_{21}^{d}(i,j;a,c,e,g,J,L)\big)+E\big(\Psi_{22}^{d}(i,j;a,c,e,g,J,L)\big)\\
&\quad=E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gi})V_{di}\Big)\\
&\qquad\quad+E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gj})V_{dj}\Big)\\
&\quad=E\Big(\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})B_L(W_{ei})B_J(W_{gi})V_{di}E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]\Big|S_i\Big]\Big)\\
&\qquad\quad+E\Big(\phi_i\zeta_{ai}B_L(W_{ei})E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_J(W_{gj})\big(V_{dj}-V_{di}+V_{di}\big)\Big|S_i\Big]\Big)\\
&\quad=-E\Big(\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})B_L(W_{ei})B_J(W_{gi})V_{di}\Big)\\
&\qquad\quad+E\Big(\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})B_L(W_{ei})B_J(W_{gj})E\Big[K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]^2\Big|W_j,S_i\Big]\Big)\\
&\qquad\quad+E\Big(\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})B_L(W_{ei})B_J(W_{gj})V_{di}E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]\Big|W_j,S_i\Big]\Big)\\
&\quad=O\big(l_n^{-1}\big)+O(b_2)E\Big(|\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})||B_L(W_{ei})|E\Big[|B_J(W_{gj})|\Big|S_i\Big]\Big)\\
&\qquad\quad+O(1)E\Big(|\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})||B_L(W_{ei})||V_{di}|E\Big[|B_J(W_{gj})\Big|S_i\Big]\Big)\\
&\quad=O(l_n^{-1})+O(l_n^{-1/2}b_2)E\Big(|\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})||B_L(W_{ei})|\Big)+O(l_n^{-1/2})E\Big(|\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})||B_L(W_{ei})||V_{di}|\Big)\\
&\quad=O(l_n^{-1})+O(l_n^{-2}b_2)+O(l_n^{-2})=O(l_n^{-1})+O(l_n^{-2}).
\end{aligned}$$

By Lemma 8.

$$\begin{aligned}
&E\big[E\big(\Psi_{21}^{d}(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}+E\big[E\big(\Psi_{22}^{d}(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}\\
&\quad=E\Big(\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2B_J(W_{gi})^2V_{di}^2E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]\Big|S_i\Big]^2\Big)^{1/2}\\
&\qquad\quad+E\Big(\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]B_J(W_{gj})\big(V_{dj}-V_{di}+V_{di}\big)\Big|S_i\Big]^2\Big)^{1/2}\\
&\quad=E\Big(\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2B_J(W_{gi})^2V_{di}^2\Big)^{1/2}\\
&\qquad\quad+E\Big(\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2\Big\{E\Big[B_J(W_{gj})E\Big(K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]^2\Big|W_i,S_i\Big)\Big|S_i\Big]\\
&\qquad\qquad\qquad+V_{di}E\Big[B_J(W_{gj})E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^{d}[b_2^{-1}(V_{dj}-V_{di})]\Big|W_i,S_i\Big)\Big|S_i\Big]\Big\}^2\Big)^{1/2}\\
&\quad=O(1)+E\Big(\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2\Big\{O(b_2)E\Big[|B_J(W_{gj})|\Big|S_i\Big]+O(1)|V_{di}|E\Big[|B_J(W_{gj})|\Big|S_i\Big]\Big\}^2\Big)^{1/2}\\
&\quad=O(1)+O(l_n^{-1/2})E\Big(\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2V_{di}^2\Big)^{1/2}\\
&\quad=O(1)+O(l_n^{-1/2}).
\end{aligned}$$

By Lemma 8.

$$E\big[E\big(\Psi_{21}^{d}(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}+E\big[E\big(\Psi_{22}^{d}(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}$$

$$
\begin{aligned}
&= E\Big(E\Big[\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gi})V_{di}\Big|S_j\Big]^2\Big)^{1/2}\\
&\quad + E\Big(E\Big[\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gj})V_{dj}\Big|S_j\Big]^2\Big)^{1/2}\\
&\le \sup_{Z_c,X,V\in G_{Z_cXV}} |\phi(X,V)\zeta(Z_c,X,V)H_2^{(1)d}(V_d)|\\
&\quad \times \Big\{E\Big(E\Big[B_L(W_{ei})B_J(W_{gi})V_{di}E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]\Big|S_j,W_i\Big)\big|S_j\Big]^2\Big)^{1/2}\\
&\quad\quad + E\Big(B_J(W_{gj})^2V_{dj}^2E\Big[B_L(W_{ei})E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]\Big|S_j,W_i\Big)\big|S_j\Big]^2\Big)^{1/2}\Big\}\\
&= O(1)E\Big(E\Big[|B_L(W_{ei})||B_J(W_{gi})|E\big(|V_{di}|\big|W_i\big)\big|S_j\Big]^2\Big)^{1/2} + O(1)E\Big(B_J(W_{gj})^2E\big(V_{dj}^2|W_j\big)E\Big[B_L(W_{ei})\Big]^2\Big)^{1/2}\\
&= O(1)E\Big(E\Big[|B_L(W_{ei})||B_J(W_{gi})|\big|S_j\Big]^2\Big)^{1/2} + O(l_n^{-1/2})E\Big(B_J(W_{gj})^2\Big)^{1/2}\\
&= O(1) + O(l_n^{-1/2}).
\end{aligned}
$$

By Lemma 8.

$$
\begin{aligned}
&E\big[E\big(\Psi_{21}^d(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2} + E\big[E\big(\Psi_{22}^d(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}\\
&= E\Big(\phi_i^2\zeta_{ai}^2b_2^{-2}K_{2ji}^{(1)}(V_d)^2[\theta_{2j}^d]^2H_2^{(1)d}(Z_{ci})^2[b_2^{-1}(V_{dj}-V_{di})]^2B_L(W_{ei})^2B_J(W_{gi})^2V_{di}^2\Big)^{1/2}\\
&\quad + E\Big(\phi_i^2\zeta_{ai}^2b_2^{-2}K_{2ji}^{(1)}(V_d)^2[\theta_{2j}^d]^2H_2^{(1)d}(Z_{ci})^2[b_2^{-1}(V_{dj}-V_{di})]^2B_L(W_{ei})^2B_J(W_{gj})^2V_{dj}^2\Big)^{1/2}\\
&\le \sup_{X,V\in G_{XV}} \theta_2^d(X,V)E\Big(\phi_i^2\zeta_{ai}^2B_L(W_{ei})^2B_J(W_{gi})^2V_{di}^2H_2^{(1)d}(Z_{ci})^2E\Big[b_2^{-2}K_{2ji}^{(1)}(V_d)^2\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^2\Big|S_i\Big]\Big)^{1/2}\\
&\quad + \sup_{X,V\in G_{XV}} \theta_2^d(X,V)\sup_{Z_c,X,V\in G_{Z_cXV}} |\phi(X,V)\zeta(Z_c,X,V)H_2^{(1)d}(V_d)|\\
&\quad\quad + E\Big(B_L(W_{ei})^2B_J(W_{gj})^2V_{dj}^2E\Big[b_2^{-2}K_{2ji}^{(1)}(V_d)^2\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]^2\Big|W_i,S_j\Big]\Big)^{1/2}\\
&= O(b_2^{-1/2})E\Big(\phi_i^2\zeta_{ai}^2B_L(W_{ei})^2B_J(W_{gi})^2V_{di}^2H_2^{(1)d}(Z_{ci})^2\Big)^{1/2} + O(b_2^{-1/2})\Big[E\Big(B_L(W_{ei})^2\Big)E\Big(B_J(W_{gj})^2E\Big[V_{dj}^2\Big|W_i\Big]\Big)\Big]^{1/2}\\
&= O(b_2^{-1/2}) + O(b_2^{-1/2})E\Big(B_J(W_{gj})^2\Big)^{1/2} = O(b_2^{-1/2}) + O(b_2^{-1/2}).
\end{aligned}
$$

By Lemma 8. Consequently,

$$
\begin{aligned}
&U_{21}^{(2)d}(a,c,e,g,J,L) + U_{22}^{(2)d}(a,c,e,g,J,L)\\
&\quad = E\big(\Psi_{21}^d(i,j;a,c,e,g,J,L)\big) + E\big(\Psi_{22}^d(i,j;a,c,e,g,J,L)\big)\\
&\quad\quad + O_p\big(n^{-1/2}E\big[E\big(\Psi_{21}^d(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi_{22}^d(i,j;a,c,e,g,J,L)|S_i\big)^2\big]^{1/2}\big)\\
&\quad\quad + O_p\big(n^{-1/2}E\big[E\big(\Psi_{21}^d(i,j;a,c,e,g,J,L)|S_j\big)^2\big]^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi_{22}^d(i,j;a,c,e,g,J,L)|S_j\big)^2\big]^{1/2}\big)\\
&\quad\quad + O_p\big(n^{-1}E\big[\Psi_{21}^d(i,j;a,c,e,g,J,L)^2\big]^{1/2}\big) + O_p\big(n^{-1}E\big[\Psi_{22}^d(i,j;a,c,e,g,J,L)^2\big]^{1/2}\big)\\
&\quad = O(l_n^{-1}) + O(l_n^{-1}) + O_p(n^{-1/2}) + O_p(l_n^{-1/2}n^{-1/2}) + O_p(n^{-1}b_2^{-1/2}) + O_p(n^{-1}b_2^{-1/2})\\
&\quad = O_p(l_n^{-1}).
\end{aligned}
$$

By Lemma 3.

$$
\begin{aligned}
E\big(\Psi_{R23}^d(i,j,t;a,c,e,g,J,L)\big) &= E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^dH_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\Big)\\
&= E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^dH_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})E\big(V_{dt}\big|W_t,S_{-t}\big)\Big)\\
&= 0.
\end{aligned}
$$

$$
E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_i\big]^2\big)^{1/2}
$$

$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\big|S_i\Big)^2\Big]^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})E\big(V_{dt}|W_t,S_{-t}\big)\big|S_i\Big)^2\Big]^{1/2}$$
$$= 0.$$

$$E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_j\big]^2\big)^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\big|S_j\Big)^2\Big]^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})E\big(V_{dt}|W_t,S_{-t}\big)\big|S_j\Big)^2\Big]^{1/2}$$
$$= 0.$$

$$E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_t\big]^2\big)^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\big|S_t\Big)^2\Big]^{1/2}$$
$$= E\Big[B_J(W_{gt})^2V_{dt}^2E\Big(\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})B_L(W_{ei})E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]\big|S_{-i}\Big]\big|S_t\Big)^2\Big]^{1/2}$$
$$= E\Big[B_J(W_{gt})^2V_{dt}^2E\Big(\phi_i\zeta_{ai}H_2^{(1)d}(Z_{ci})B_L(W_{ei})\big|S_t\Big)^2\Big]^{1/2}$$
$$= O\big(l_n^{-3/2}\big)E\Big[B_J(W_{gt})^2E\big(V_{dt}^2|W_t\big)\Big]^{1/2} = O\big(l_n^{-3/2}\big)E\Big[B_J(W_{gt})^2\Big]^{1/2} = O\big(l_n^{-3/2}\big).$$

By Lemma 8

$$E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_{-i}\big]^2\big)^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\big|S_{-i}\Big)^2\Big]^{1/2}$$
$$\le \sup_{X,V\in G_{XV}}\theta_2^d(X,V)\sup_{Z_c,X,V\in G_{Z_cXV}}|\phi(X,V)\zeta(Z_c,X,V)H_2^{(1)d}(V_d)|$$
$$\times E\Big[B_J(W_{gt})^2E\big(V_{dt}^2|W_t,S_{-t}\big)E\Big(|B_L(W_{ei})|E\Big[b_2^{-1}K_{2ji}^{(1)}(V_d)[b_2^{-1}(V_{dj}-V_{di})]\Big|W_i,S_{-i}\Big]\Big|S_{-i}\Big)^2\Big]^{1/2}$$
$$= O(1)E\Big[B_J(W_{gt})^2E\Big(|B_L(W_{ei})|\Big|S_{-i}\Big)^2\Big]^{1/2} = O(l_n^{-1/2})E\big[B_J(W_{gt})^2\big]^{1/2} = O(l_n^{-1/2}).$$

$$E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_{-j}\big]^2\big)^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\big|S_{-j}\Big)^2\Big]^{1/2}$$
$$= E\Big[\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2B_J(W_{gt})^2V_{dt}^2E\Big(b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d[b_2^{-1}(V_{dj}-V_{di})]\big|S_{-j}\Big)^2\Big]^{1/2}$$
$$= E\Big[\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2B_J(W_{gt})^2E\big(V_{dt}^2\big|W_t,S_{-t}\big)\Big]^{1/2}$$
$$= E\Big[\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2E\big(B_J(W_{gt})^2\big|S_{-t}\big)\Big]^{1/2}$$
$$= E\Big[\phi_i^2\zeta_{ai}^2H_2^{(1)d}(Z_{ci})^2B_L(W_{ei})^2\Big]^{1/2} = O(l_n^{-1/2}).$$

By Lemma 8.

$$E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_{-t}\big]^2\big)^{1/2}$$

$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})V_{dt}\big|S_{-t}\Big)^2\Big]^{1/2}$$
$$= E\Big[E\Big(\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(1)}(V_d)\theta_{2j}^d H_2^{(1)d}(Z_{ci})[b_2^{-1}(V_{dj}-V_{di})]B_L(W_{ei})B_J(W_{gt})E\big(V_{dt}\big|W_t,S_{-t}\big)\big|S_{-t}\Big)^2\Big]^{1/2}$$
$$= 0.$$

$$E\big[\Psi_{23}^d(i,j,t;a,c,e,g,J,L)^2\big]^{1/2} = E\Big(\phi_i^2\zeta_{ai}^2 b_2^{-2}K_{2ji}^{(1)}(V_d)^2[\theta_{2j}^d]^2 H_2^{(1)d}(Z_{ci})^2[b_2^{-1}(V_{dj}-V_{di})]^2 B_L(W_{ei})^2 B_J(W_{gt})^2 V_{dt}^2\Big)$$
$$\le \sup_{X,V\in G_{XV}} \theta_2^d(X,V)^{1/2}$$
$$\times E\Big(\phi_i^2\zeta_{ai}^2 H_2^{(1)d}(Z_{ci})^2 B_L(W_{ei})^2 E\Big[b_2^{-2}K_{2ji}^{(1)}(V_d)^2[\theta_{2j}^d]^2[b_2^{-1}(V_{dj}-V_{di})]^2\Big|S_{-j}\Big]E\Big[B_J(W_{gt})^2 E\big(V_{dt}^2\big|W_t,S_{-t}\big)\Big|S_{-t}\Big]\Big)$$
$$= O(b_2^{-1/2})E\Big(\phi_i^2\zeta_{ai}^2 H_2^{(1)d}(Z_{ci})^2 B_L(W_{ei})^2 E\Big[B_J(W_{gt})^2\Big|S_{-t}\Big]\Big)^{1/2}$$
$$= O(b_2^{-1/2})E\Big(\phi_i^2\zeta_{ai}^2 H_2^{(1)d}(Z_{ci})^2 B_L(W_{ei})^2\Big)^{1/2} = O(l_n^{-1/2}b_2^{-1/2}).$$

By Lemma 8. As a result.

$$U_{23}^{(3)d}(a,c,e,g,J,L)$$
$$= E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_i\big]^2\big)^{1/2}\big)$$
$$+ O_p\big(n^{-1/2}E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_j\big]^2\big)^{1/2}\big) + O_p\big(n^{-1/2}E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_t\big]^2\big)^{1/2}\big)$$
$$+ O_p\big(n^{-1}E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_{-i}\big]^2\big)^{1/2}\big) + O_p\big(n^{-1}E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_{-j}\big]^2\big)^{1/2}\big)$$
$$+ O_p\big(n^{-1}E\big[E\big(\Psi_{23}^d(i,j,t;a,c,e,g,J,L)\big|S_{-t}\big]^2\big)^{1/2}\big) + O_p\big(n^{-3/2}E\big[\Psi_{23}^d(i,j,t;a,c,e,g,J,L)^2\big]^{1/2}\big)$$
$$= O_p(l_n^{-3/2}n^{-1/2}) + O_p(n^{-1}l_n^{-1/2}) + O_p(n^{-3/2}l_n^{-1/2}b_2^{-1/2})$$
$$= O_p(l_n^{-3/2}n^{-1/2}).$$

In summary,

$$R_{2dc}(a) = \sum_{e,g}^{q}\sum_{L,J}^{l_n+2k} Q_{BB}^{-1}(e,g,L,J)\Big\{n^{-1}U_{21}^{(2)d}(a,c,e,g,J,L) + n^{-1}U_{22}^{(2)d}(a,c,e,g,J,L) + U_{23}^{(3)d}(a,c,e,g,J,L)\Big\}$$
$$\le 1'_{q(l_n+2k)}Q_{BB}^{-1}1_{q(l_n+2k)} \max_{\substack{1\le e,g\le q\\ 1\le J,L\le l_n+2k}} \Big\{n^{-1}U_{21}^{(2)d}(a,c,e,g,J,L) + n^{-1}U_{22}^{(2)d}(a,c,e,g,J,L) + U_{23}^{(3)d}(a,c,e,g,J,L)\Big\}$$
$$\le q(l_n+2k)\big[\lambda_{\min}\big(Q_{BB}\big)\big]^{-1}\big\{O_p\big(n^{-1}l_n^{-1}\big) + O_p\big(n^{-1/2}l_n^{-3/2}\big)\big\}$$
$$= O_p(n^{-1/2}l_n^{-1/2}) = o_p(n^{-1/2}).$$

by assumption A1. Futhermore,

$$B_{31233dc}(a) = B_{31233dc}^*(a) + o_p(n^{-1/2})$$
$$= R_{1dc}(a) - R_{2dc}(a) + R_{3dc}(a) - R_{4dc}(a) + R_{5dc}(a) + o_p(n^{-1/2})$$
$$= o_p(n^{-1/2}),$$

and,

$$B_{3123dc}(a) = B_{31231dc}(a) + B_{31232dc}(a) + B_{31233dc}(a) = o_p(n^{-1/2}).$$

$B_{3124dc}(a) + B_{3125dc}(c) + B_{3126dc}(a)$

$$
\begin{aligned}
&= [2n(n-1)b_2^2]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(2)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^2\mathsf{C}_{2ji}^d(c)\\
&\quad+[6n(n-1)b_2^3]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(3)}(V_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^3\mathsf{C}_{2ji}^d(c)\\
&\quad+[24n(n-1)b_2^5]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\phi_i\zeta_{ai}K_{2ji}^{(4)}(\tilde{V}_d)[\hat{m}_d^{l_n}(W_i)-m_d(W_i)-\big(\hat{m}_d^{l_n}(W_j)-m_d(W_j)\big)\big]^4\mathsf{C}_{2ji}^d(c)\\
&\leq \big[2b_2^{-1}\sup_{W\in G_W}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|\big]^2[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\big|\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(2)}(V_d)\mathsf{C}_{2ji}^d(c)\big|\\
&\quad+\big[2b_2^{-1}\sup_{W\in G_W}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|\big]^3[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\big|\phi_i\zeta_{ai}b_2^{-1}K_{2ji}^{(3)}(V_d)\mathsf{C}_{2ji}^d(c)\big|\\
&\quad+\big[2b_2^{-1}\sup_{W\in G_W}\big|\hat{m}_d^{l_n}(W)-m_d(W)\big|\big]^3b_2^{-1}[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\big|\phi_i\zeta_{ai}K_{2ji}^{(4)}(\tilde{V}_d)\mathsf{C}_{2ji}^d(c)\big|\\
&= O_p\Big([L_nb_2^{-1}]^2\Big)[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_iu_i|\big|b_2^{-1}K_{1ji}^{(2)}(V_d)\big|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\quad+O_p\Big([L_nb_2^{-1}]^3\Big)[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_iu_i|\big|b_2^{-1}K_{1ji}^{(3)}(V_d)\big|\big|\mathsf{C}_{2ji}^d(c)\big|\\
&\quad+O_p\Big([L_n^4b_2^{-5}]\Big)[n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}|\phi_iu_i|\big|\mathsf{C}_{2ji}^d(c)\big|.
\end{aligned}
$$

Note that the proof of the order of $B_{3124dc}(a)+B_{3125dc}(c)+B_{3126dc}(a)$ is, mutatis mutandis, (exchanging u_i for ζ_{ai}) practically identical to the proof of the order of $B_{1424d}(c)+B_{1425d}(c)+B_{1426d}(c)$. Thus the arguments will not be repeated here. As a result, one can conclude,

$$B_{3124dc}(a)+B_{3125dc}(c)+B_{3126dc}(a)=o_p(n^{-1/2}).$$

Furthermore,

$$B_{312dc}=B_{3121dc}+B_{3122dc}+B_{3123dc}+B_{3124dc}+B_{3125dc}(c)+B_{3126dc}=o_p(n^{-1/2}).$$

Note that the proof of the order of B_{311d} is, mutatis mutandis, (exchanging Z_{ci} for Y_i) practically identical to the proof of the order of B B_{312dc}. Thus will not be repeated here. As a result, one can conclude,$B_{311d}=o_p(n^{-1/2})$. Consequently,

$$B_{31d}=B_{311d}-\sum_{c=1}^{p}\beta_{1c}B_{312dc}\leq o_p(n^{-1/2})\big[1+\sum_{c=1}^{p}|\beta_{1c}|\big]=o_p(n^{-1/2})\big[1+||\beta_1||_E\big]=o_p(n^{-1/2}).$$

Also, $B_3=\sum_{d=1}^{d}\big[B_{31d}+B_{32d}+B_{33d}+B_{34d}\big]=o_p(n^{-1/2})$. Now note that,

$$
\begin{aligned}
B_4&=(2D-1)n^{-1}\big[\mathbf{Z}_n-\hat{\mathbf{H}}_n^*(Z)\big]'\hat{\boldsymbol{\phi}}_n\Big(\big[\boldsymbol{\mu}_Z-\hat{\boldsymbol{\mu}}_Z\big]\beta_1-\big[\boldsymbol{\mu}_Y-\hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&=(2D-1)n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\Big(\big[\boldsymbol{\mu}_Z-\hat{\boldsymbol{\mu}}_Z\big]\beta_1-\big[\boldsymbol{\mu}_Y-\hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&\quad+(2D-1)n^{-1}\big[\mathbf{Z}_n-\mathbf{H}_n^*(Z)\big]'(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n)\Big(\big[\boldsymbol{\mu}_Z-\hat{\boldsymbol{\mu}}_Z\big]\beta_1-\big[\boldsymbol{\mu}_Y-\hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&\quad+(2D-1)n^{-1}\big[\mathbf{H}_n^*(Z)-\hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\Big(\big[\boldsymbol{\mu}_Z-\hat{\boldsymbol{\mu}}_Z\big]\beta_1-\big[\boldsymbol{\mu}_Y-\hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&\quad+(2D-1)n^{-1}\big[\mathbf{H}_n^*(Z)-\hat{\mathbf{H}}_n^*(Z)\big]'(\hat{\boldsymbol{\phi}}_n-\boldsymbol{\phi}_n)\Big(\big[\boldsymbol{\mu}_Z-\hat{\boldsymbol{\mu}}_Z\big]\beta_1-\big[\boldsymbol{\mu}_Y-\hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&\equiv(2D+1)\big[B_{41}+B_{42}+B_{43}+B_{44}\big].
\end{aligned}
$$

Note that by Lemma 2,

$$E\Big(\Big|n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ci}\Big|^2\Big) = n^{-2}\sum_{i=1}^{n}E\big(\phi^2\zeta_{ci}^2\big) + [n(n-1)]^{-1}\sum_{i=1}^{n}\sum_{j\neq i}E\big(\phi_i\zeta_{ci}\big)E\big(\phi_j\zeta_{cj}\big) = O(n^{-1}).$$

Consequently by Markov's Inequality, $|n^{-1}\sum_{i=1}^{n}\phi_i\zeta_{ci}\big| = O_p(n^{-1/2})$.

$$\begin{aligned}
B_{41} &= n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\boldsymbol{\phi}_n\Big(\big[\boldsymbol{\mu}_Z - \hat{\boldsymbol{\mu}}_Z\big]\beta_1 - \big[\boldsymbol{\mu}_Y - \hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i - H^*(Z_i)\big]\phi_i\Big(\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big)\\
&\leq \Big|\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big|\Big|n^{-1}\sum_{i=1}^{n}\phi_i\zeta_i\Big|\\
&= O_p(\mathcal{L}_{0n})O_p(n^{-1/2})\big(1 + ||\beta_1||_E\big) = o_p(n^{-1/2}).
\end{aligned}$$

By Theorem 2 and Markov's Inequality,

$$\begin{aligned}
B_{42} &= n^{-1}\big[\mathbf{Z}_n - \mathbf{H}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\Big(\big[\boldsymbol{\mu}_Z - \hat{\boldsymbol{\mu}}_Z\big]\beta_1 - \big[\boldsymbol{\mu}_Y - \hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&= n^{-1}\sum_{i=1}^{n}\big[Z_i - H^*(Z_i)\big]\big(\hat{\phi}_i - \phi_i\big)\Big(\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big)\\
&\leq \Big|\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big|\sup_{X,V\in G_{XV}}\big|\hat{\phi}(X,\hat{V}) - \phi(X,V)\big|n^{-1}\sum_{i=1}^{n}|\zeta_i|\\
&= O_p(\mathcal{L}_{0n}^2)O_p(1)\big(1 + ||\beta_1||_E\big) = o_p(n^{-1/2}).
\end{aligned}$$

By Theorem 1 and 2, and Markov's Inequality,

$$\begin{aligned}
B_{43} &= n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\boldsymbol{\phi}_n\Big(\big[\boldsymbol{\mu}_Z - \hat{\boldsymbol{\mu}}_Z\big]\beta_1 - \big[\boldsymbol{\mu}_Y - \hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&= n^{-1}\sum_{i=1}^{n}\big[H^*(Z_i) - \hat{H}^*(Z_i)\big]\phi_i\Big(\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big)\\
&\leq \Big|\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big|\sup_{X,V\in G_{X,V}}\big|H^*(Z_i) - \hat{H}^*(Z_i)\big|n^{-1}\sum_{i=1}^{n}|\phi_i|\\
&O_p(\mathcal{L}_{0n})O_p(\mathcal{L}_n)\big(1 + ||\beta_1||_E\big) = o_p(n^{-1/2}),
\end{aligned}$$

By Theorem 2 and Markov's Inequality,

$$\begin{aligned}
B_{44} &= n^{-1}\big[\mathbf{H}_n^*(Z) - \hat{\mathbf{H}}_n^*(Z)\big]'\big(\hat{\boldsymbol{\phi}}_n - \boldsymbol{\phi}_n\big)\Big(\big[\boldsymbol{\mu}_Z - \hat{\boldsymbol{\mu}}_Z\big]\beta_1 - \big[\boldsymbol{\mu}_Y - \hat{\boldsymbol{\mu}}_Y\big]\Big)\\
&= n^{-1}\sum_{i=1}^{n}\big[H^*(Z_i) - \hat{H}^*(Z_i)\big]\big(\hat{\phi}_i - \phi_i\big)\Big(\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big)\\
&\leq \Big|\sum_{j=1}^{p}\big[\mu_{Z_j} - \hat{\mu}_{Z_j}\big]\beta_{1j} - [\mu_Y - \hat{\mu}_Y]\Big|\sup_{X,V\in G_{XV}}\big|\hat{\phi}(X,\hat{V}) - \phi(X,V)\big|\sup_{X,V\in G_{X,V}}\big|H^*(Z_i) - \hat{H}^*(Z_i)\big|\\
&= O_p(\mathcal{L}_{0n}^2)O_p(\mathcal{L}_n)\big(1 + ||\beta_1||_E\big) = o_p(n^{-1/2}),
\end{aligned}$$

by Theorem 1 and 2. Consequently,

$$B_4 = (2D-1)\big[B_{41} + B_{42} + B_{43} + B_{44}\big] = o_p(n^{-1/2}).$$

In summary,

$$\sqrt{n}B = \sqrt{n}B_1 + \sqrt{n}B_2 + \sqrt{n}B_3 + \sqrt{n}B_4 = \sqrt{n}B_1 + o_p(1)$$

Hence, by the Cramer Wold Device,

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = A^{-1}\sqrt{n}B \xrightarrow{d} N(0, \Sigma_0^{-1}\Sigma_1\Sigma_0^{-1}).$$

$\square$

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